Journal of Computational Physics 322 (2016) 309–325

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

Boundary Variation Diminishing (BVD) reconstruction: A new approach to improve Godunov schemes

Ziyao Sun, Satoshi Inaba, Feng Xiao *

Department of Mechanical Engineering, Tokyo Institute of Technology, 4259 Nagatsuta Midori-ku, Yokohama, 226-8502, Japan

ARTICLE INFO

Article history:
Received 4 February 2016
Received in revised form 28 June 2016
Accepted 28 June 2016
Available online 5 July 2016

Keywords:
Finite volume method
Godunov method
Discontinuity
Reconstruction
Boundary Variation Diminishing (BVD)
WENO
THINC
Riemann solver
Compressible flow
Shock wave

ABSTRACT

This paper presents a new approach, so-called boundary variation diminishing (BVD), for reconstructions that minimize the discontinuities (jumps) at cell interfaces in Godunov type schemes. It is motivated by the observation that diminishing the jump at the cell boundary can effectively reduce the dissipation in numerical flux. Differently from the existing practices which seek high-order polynomials within mesh cells while assuming discontinuities being always at the cell interfaces, the BVD strategy presented in this paper switches between a high-order polynomial and a jump-like reconstruction that allows a discontinuity being partly represented within the mesh cell rather than at the interface. Excellent numerical results have been obtained for both scalar and Euler conservation laws with substantially improved solution quality in comparison with the existing methods. It is shown that new schemes of high fidelity for both continuous and discontinuous solutions can be devised by the BVD guideline with properly-chosen candidate reconstruction schemes. This work provides a simple and accurate alternative of great practical significance to the current high-order Godunov paradigm which overly pursues the smoothness within mesh cells under the questionable premiss that discontinuities only appear at cell interfaces.

1. Introduction

The high-resolution shock capturing schemes, which are now well-accepted as the main-stream numerical approach to solve hyperbolic conservation laws including the Euler equations for compressible gas dynamics, trace back to Godunov's scheme [13]. The original Godunov scheme is a conservative finite-volume method (FVM) with piece-wise constant reconstruction, and the fluxes at cell boundaries are computed by solving the exact Riemann problem given the piece-wise constant physical fields in the two neighboring cells which usually results in a jump at cell boundaries. In general, Godunov schemes consist of two essential steps in solution procedure, i.e. (I) reconstruct the physical fields to find the values at the left- and right- sides of cell boundaries, and (II) evaluate the numerical fluxes at cell boundaries that are needed in the FVM formulation to update the cell-integrated values for next time step.

For step (II), interested readers are referred to [38] for a monographic review. The main interest of this paper is limited to step (I), i.e. reconstruction. Continuous efforts have been devoted in the past half century to develop the original Godunov scheme into a class of high-resolution conservative schemes. Instead of the piece-wise constant approximation which leads to a first-order scheme, higher order polynomials have been used to improve the accuracy (convergence rate for smooth

* Corresponding author.
E-mail address: xiao.f.aa@m.titech.ac.jp (F. Xiao).

http://dx.doi.org/10.1016/j.jcp.2016.06.051

solution). In order to get around the so-called Godunov barrier for linear schemes which states that any monotonic linear scheme can be of only first order, non-linear numerical dissipation has been introduced to high-order schemes to suppress spurious oscillations in the vicinity of discontinuities. The non-linear numerical dissipation has been devised in different forms seen in the literature as either flux limiters in flux-corrected transport (FCT) scheme [5,44] and total variation diminishing (TVD) scheme [14,36] or slope limiters in monotone upstream-centered schemes for conservation law (MUSCL) scheme [39,40]. The later gives a more straightforward interpretation of the polynomial-based reconstruction for high-order schemes, and shows a clear path to design high-resolution schemes using modified high-order polynomials. Representative schemes of this kind are piecewise parabolic method (PPM) method [9], essentially non-oscillatory scheme (ENO) [15,32,33,12] and weighted essentially non-oscillatory (WENO) scheme [21,19,31]. WENO concept provides a general framework to develop high-resolution schemes that are able to reach the highest possible accuracy over a given mesh stencil for smooth solution, and effectively stabilize the numerical solutions which include discontinuities. Successive works on WENO scheme are found in designing better smoothness indicators and nonlinear weights [16,4,3,30,17,20] to improve solution quality, as well as implementing the WENO concept to different discretization frameworks [10,25,28,22].

Starting from the piecewise constant approximation in the original Godunov scheme, the development of high-order shock capturing schemes has evidenced a history of efforts to reduce the difference between the reconstructed left- and right-side values at cell boundaries for smooth solutions. This observation, however to our knowledge, has not been paid enough attention, nor explored further as a prospective guideline to construct new schemes.

In this work, we propose a new guideline to design high-resolution schemes, called boundary variation diminishing (BVD) reconstruction. The BVD reconstruction shows consistent results with the observation gained for smooth solutions as mentioned above. More importantly, it also provides a principle to devise the reconstruction for discontinuity. We demonstrate in this paper the BVD reconstruction based on some existing schemes, i.e. the 5th-order WENO-Z method [4], the 5th-order targeted ENO (TENO5) method [12] and the THINC (Tangent of Hyperbola for INterface Capturing) method [42,43,34].

This paper is composed as follows. We present the BVD principle in Section 2, as well as the reconstruction schemes that fit separately smooth and discontinuous profiles as the building blocks for the BVD algorithm. The BVD-based numerical methods are then extensively verified in Section 3 with widely used benchmark tests, which demonstrate the significance and impact of the presented methods in resolving both smooth and discontinuous solutions. The paper ends with a brief summary in Section 4.

2. Numerical algorithm

We use a scalar conservation law in the following form to present the BVD algorithm

$$\frac{\partial u}{\partial t}+\frac{\partial f}{\partial x}=0, \tag{1}$$

where $u(x,t)$ is the solution function and $f(u)$ the flux function. For a hyperbolic equation, $\alpha = f'(u)$ is a real number, the characteristic speed.

We divide the computational domain into N non-overlapping cell elements, $I_i=[x_{i-\frac{1}{2}},x_{i+\frac{1}{2}}]$, $i=1,2,\cdots,N$. The mesh is assumed to be uniform across the computational domain for simplicity, $\Delta x = x_{i+\frac{1}{2}}-x_{i-\frac{1}{2}}$, which is not essential and formulations on non-uniform meshes can be constructed under the same concept.

Using an FVM framework, we define the volume-integrated average (VIA) of the solution function $u(x,t)$ for cell I_i as

$$\overline{u}_i(t)=\frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x,t)dx. \tag{2}$$

The VIA $\bar{u}_i$ of each cell $I_i=[x_{i-\frac{1}{2}},x_{i+\frac{1}{2}}]$ is updated by

$$\frac{d\bar{u}_i}{dt}=-\frac{1}{\Delta x}(\tilde{f}_{i+\frac{1}{2}}-\tilde{f}_{i-\frac{1}{2}}), \tag{3}$$

where the numerical fluxes at cell boundaries are computed by a Riemann solver

$$\tilde{f}_{i+\frac{1}{2}}=f^{\text{Riemann}}_{i+\frac{1}{2}}(u^L_{i+\frac{1}{2}},u^R_{i+\frac{1}{2}}), \tag{4}$$

using the left-side value u^L and right-side value u^R obtained from the reconstructions over left- and right-biased stencils. In spite of different variants, the Riemann flux is essentially upwinding and can be thus written in a canonical form as

$$f^{\text{Riemann}}_{i+\frac{1}{2}}(u^L_{i+\frac{1}{2}},u^R_{i+\frac{1}{2}})=\frac{1}{2}\left(f(u^L_{i+\frac{1}{2}})+f(u^R_{i+\frac{1}{2}})-|\tilde{\alpha}_{i+\frac{1}{2}}|(u^R_{i+\frac{1}{2}}-u^L_{i+\frac{1}{2}})\right), \tag{5}$$