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Constrained hyperbolic divergence cleaning in smoothed particle magnetohydrodynamics with variable cleaning speeds

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ABSTRACT

We present an updated constrained hyperbolic/parabolic divergence cleaning algorithm for smoothed particle magnetohydrodynamics (SPMHD) that remains conservative with wave cleaning speeds which vary in space and time. This is accomplished by evolving the quantity ψ/c_{h} instead of ψ. Doing so allows each particle to carry an individual wave cleaning speed, c_{h}, that can evolve in time without needing an explicit prescription for how it should evolve, preventing circumstances which we demonstrate could lead to runaway energy growth related to variable wave cleaning speeds. This modification requires only a minor adjustment to the cleaning equations and is trivial to adopt in existing codes. Finally, we demonstrate that our constrained hyperbolic/parabolic divergence cleaning algorithm, run for a large number of iterations, can reduce the divergence of the magnetic field to an arbitrarily small value, achieving $\nabla \cdot \mathbf{B} = 0$ to machine precision.

1. Introduction

Accurately evolving the equations of magnetohydrodynamics (MHD) in numerical simulations is crucial in astrophysical fluid dynamics. In smoothed particle magnetohydrodynamics (SPMHD) [14,25,32–34,28], upholding the divergence-free constraint of the magnetic field has been the main technical difficulty. The usual approach is to evolve the magnetic field directly by the induction equation (as in [25]), but this preserves a divergence-free magnetic field only to truncation error. These errors cause more harm than just yielding an unphysical field. They introduce spurious monopole accelerations, which have to be carefully handled in SPMHD in order to ensure numerical stability, at the price of no longer exactly conserving momentum [25,22,3]. Handling the divergence-free constraint on the magnetic field is therefore one of the most important aspects of accurate SPMHD simulations.

One option is to define the magnetic field in a way that manifestly enforces the divergence-free constraint. Use of the Euler potentials, $\mathbf{B} = \nabla\alpha \times \nabla\beta$ where α and β are passive scalars, was proposed as early as Phillips and Monaghan [25], and recently the potentials have been used for simulations of protostar formation [29], star cluster formation [30,31] and magnetised galaxies [12,17]. However, the Euler potentials cannot represent winding motions, prevent dynamo processes by construction [6], and it is not clear how to incorporate non-ideal dissipation. A vector potential implementation, $\mathbf{B} = \nabla \times \mathbf{A}$, was tested for SPMHD by Price [27], but was found to be numerically unstable. Stasyszyn and Elstner [37] recently proposed that the vector potential could be used, if one added numerical diffusion to the potential, enforced the Coulomb gauge

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condition on the vector potential ($\nabla \cdot \mathbf{A} = 0$) and smoothed the resulting magnetic field, though it is not clear how robust this approach is in practice.

The second option to handle the divergence-free constraint in SPMHD is to directly evolve the magnetic field with the induction equation, but then 'clean' errors out of the field. For example, parabolic diffusion terms can be used to smooth the magnetic field at the resolution scale [22]. The artificial resistivity formulation of Price and Monaghan [32,34] has been used for this purpose [e.g., [8]], however, artificial resistivity is intended for shock capturing and dissipates physical as well as unphysical components of the field. A similar idea is to periodically smooth the magnetic field to remove fluctuations below the resolution limit [3], but this adds computational expense, is time resolution dependent, and reduces the spatial resolution of the magnetic field.

At present, the best option for divergence cleaning in SPMHD is the 'constrained' hyperbolic/parabolic divergence cleaning method of Tricco and Price [41], an improved version of the method by Dedner et al. [10]. The original idea from Dedner et al. [10] was to couple an additional scalar field, ψ, to the induction equation according to

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \nabla \psi, \tag{1}$$

$$\frac{\partial \psi}{\partial t} = -c_{\mathrm{h}}^2 (\nabla \cdot \mathbf{B}) - \frac{\psi}{\tau}, \tag{2}$$

where $\mathbf{B}$ is the magnetic field and $\mathbf{v}$ is the velocity. These may be combined to produce a damped wave equation for the divergence of the magnetic field,

$$\frac{\partial^2 (\nabla \cdot \mathbf{B})}{\partial t^2} - c_{\mathrm{h}}^2 \nabla^2 (\nabla \cdot \mathbf{B}) + \frac{1}{\tau} \frac{\partial (\nabla \cdot \mathbf{B})}{\partial t} = 0. \tag{3}$$

From Equation (3), we see that Equation (1) and the first term on the right hand side of Equation (2) represent hyperbolic transport of divergence errors at a characteristic speed, c_{h}, which we refer to as the 'wave cleaning speed'. This is typically chosen to be the fast MHD wave speed so that it obeys the local Courant condition and does not impose any additional timestep constraint. The second term on the right hand side of Equation (2) produces parabolic diffusion on a timescale defined according to

$$\tau \equiv \frac{h}{\sigma c_{\mathrm{h}}}, \tag{4}$$

where h is the smoothing length (resolution scale) and σ is a dimensionless constant with empirically determined optimal values of 0.3 and 1.0 in 2D and 3D, respectively [41]. The combination of hyperbolic and parabolic terms in Equations (1)–(2) spreads the divergence of the magnetic field over a larger area, reducing the impact of any single large source of error, while also allowing the diffusion to be more effective.

In Tricco and Price [41], we showed that the original Dedner et al. [10] approach could be unstable at density jumps and free surfaces, leading to exponential growth of magnetic energy. To remedy this, we derived a version of the cleaning equations under the constraint that the hyperbolic transport should conserve energy. Though ψ is not a physical variable, conservation of energy for the hyperbolic term between the magnetic and ψ fields ensures that, when the parabolic term is included, magnetic energy can only ever be removed by divergence cleaning, never added, guaranteeing numerical stability. The 'constrained' or 'conservative' cleaning equations we derived in Tricco and Price [41] are given by

$$\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t} = (\mathbf{B} \cdot \nabla)\mathbf{v} - \mathbf{B}(\nabla \cdot \mathbf{v}) - \nabla \psi, \tag{5}$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = -c_{\mathrm{h}}^2 (\nabla \cdot \mathbf{B}) - \frac{\psi}{\tau} - \frac{1}{2}\psi(\nabla \cdot \mathbf{v}), \tag{6}$$

where $\mathrm{d}/\mathrm{d}t \equiv \partial/\partial t + \mathbf{v} \cdot \nabla$ is the Lagrangian time derivative. The formulation of the induction equation (Equation (5)) in the absence of the $\nabla\psi$ term follows the 'divergence preserving scheme' of Powell et al. [26] (see also [16,11]), meaning that divergence errors are preserved by the flow in the absence of cleaning. The third term in Equation (6) was introduced by Tricco and Price [41] to account for changes in ψ from compression or rarefaction of the gas, and is necessary to ensure total energy conservation in the absence of damping. The practical advantage of this algorithm for SPMHD is that it adds no additional timestep constraint, is simple to implement, computationally efficient, and has been successfully used to enforce the divergence-free constraint in simulations of jets and outflows during protostar formation [35,2,18,45]. However, our original method was derived assuming that the cleaning speed, c_{h}, is constant in both space and time, but this is not true in practice and presents a source of non-conservation of energy. Furthermore, source terms are added to the right hand side of Equation (3) when c_{h} or τ are time or spatially variable, by the addition of the $\frac{1}{2}\psi(\nabla \cdot \mathbf{v})$ term, and by solving the cleaning equations in the Lagrangian frame of motion. How these source terms change the propagation of divergence errors is not properly understood, but will be addressed in this work.

In this paper, we derive an improvement to constrained hyperbolic/parabolic divergence cleaning such that the hyperbolic evolution equations remain conservative even in the presence of a variable cleaning speed (Section 2). We demonstrate that these equations create a generalised wave equation which naturally incorporates the source terms (Section 2.7). Aspects