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Exponential convergence through linear finite element discretization of stratified subdomains

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ABSTRACT

Motivated by problems where the response is needed at select localized regions in a large computational domain, we devise a novel finite element discretization that results in exponential convergence at pre-selected points. The key features of the discretization are (a) use of midpoint integration to evaluate the contribution matrices, and (b) an unconventional mapping of the mesh into complex space. Named complex-length finite element method (CFEM), the technique is linked to Padé approximants that provide exponential convergence of the Dirichlet-to-Neumann maps and thus the solution at specified points in the domain. Exponential convergence facilitates drastic reduction in the number of elements. This, combined with sparse computation associated with linear finite elements, results in significant reduction in the computational cost. The paper presents the basic ideas of the method as well as illustration of its effectiveness for a variety of problems involving Laplace, Helmholtz and elastodynamics equations.

1. Introduction

Conventional domain-based methods such as finite element and finite difference techniques obtain the solution over the entire domain. While such approaches are appropriate for many problems, there are several situations where the response is needed only in a few small regions of interest. Some examples include: (a) reservoir modeling where the response at injection and production wells are of utmost importance, (b) forward modeling in the context of nondestructive testing and system identification, where the goal is to match the response of the system at sparse discrete points in the domain and (c) structural acoustics, where the acoustic signature is not needed at all the points, but at distinct locations in the far field. Most of these problems involve significant computational expense and it would be desirable to reduce the computational cost, if it can come at the expense of not computing the response in the rest of the domain. With this motivation, this paper presents an unconventional finite element method that provides high accuracy at prescribed points in the domain. In this paper, we treat the special but important class of problems involving large regular (e.g. layered) subdomains, where the actual solution inside these subdomains may not be of interest, but it is important to capture the effect of these subdomains on the solution in the remainder of domain. Specifically, we show that a special mesh with midpoint-integrated linear finite elements results in exponential convergence of the solution on the edges of layered subdomains.

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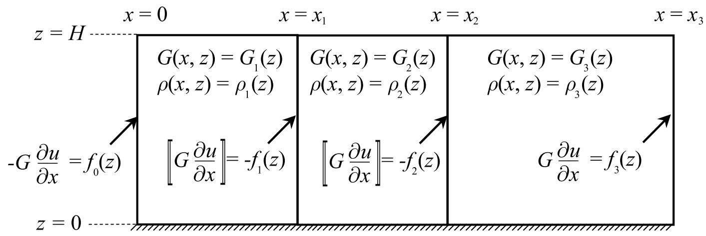

Fig. 1. A schematic of the model problem. An elastic layer is deforming out of plane under external excitation $f_{0,1,2,3}$ at the vertical interfaces. The segments bounded by these interfaces can be stratified in the vertical direction. The goal is to obtain the responses at the interfaces.

Exponential convergence is typically achieved with the help of spectral methods where the field variable is discretized with Fourier basis [1,2], but these methods typically render the computation global. On the other hand, regular finite element and finite difference methods involve more efficient sparse computation, but the convergence is only algebraic. It was discovered that exponential convergence can be obtained with sparse computation, provided that the solution is needed only at specific points in the domain [3,4]. By linking finite-difference approximation to rational approximation of the Dirichlet-to-Neumann (DtN) map, exponential convergence is achieved at the edges of sub-domains discretized with specially devised finite difference grids. The basic idea is to obtain optimal rational approximation of the *one-sided* DtN map (with Dirichlet or Neumann condition applied on the other edge), and translate the approximation to an equivalent finite difference grid. Since the grids result from exponentially convergent optimal approximations of the DtN map, they are called optimal finite difference grids and result in exponential convergence at the edges of the sub-domains. The main limitation of this method is that two distinct finite difference grids are needed, one when Dirichlet condition is applied on the other edge, and another for Neumann condition, indicating that the grids cannot be used directly for *two-sided* problems, which would be the building block for multi-domain problems. Optimal grids can be used for two-sided problems indirectly, through splitting the solution into odd and even parts, devising two separate grids for each part, and using them in a completely overlapping fashion. This idea is extended to multiple dimensions, but the computation becomes rather cumbersome, requiring increasing number of overlapping grids (four for two-dimensional problems and eight for three-dimensions) [4–6].

In this paper, we introduce a simpler alternative to optimal finite difference grids and eliminate the need for overlapping grids. The key to the proposed method is the observation that *midpoint integrated linear finite elements* preserve the DtN map of the half-space, i.e. adding these elements to a half-space does not alter the DtN map of the composite half-space [7–9]; given the equivalence between impedance and half-space DtN map, we call this property the *impedance-preserving property*. We show that this property eliminates the need for multiple overlapping grids and provides *exponential convergence of the DtN map for the two-sided problem with a single grid*. This makes the implementation attractive and the computation can be performed by a simple modification of existing finite element codes. The only complication is that the finite element mesh needs to be bent out of the real space, making the element lengths complex-valued. This feature necessitates complex arithmetic and could contribute to an increase in the computational cost. However, this increase is not an issue as the proposed method needs a very coarse grid, with number of elements typically less by an order of magnitude than that for regular finite element discretization. Moreover, in many cases including time-harmonic wave propagation, the original computation involves complex arithmetic and the mapping of the mesh into complex space does not add any further complications.

This paper focuses on the derivation of the method along with numerical examples to illustrate its efficiency. We start with the two-dimensional model problem given in Fig. 1 and show that it can be reduced to a set of one-dimensional two-point boundary value problems through semi-discretization in the vertical direction (Section 2). We then obtain a grid that simultaneously approximates the DtN map for all of the one-dimensional problems (Section 3). This is achieved by reformulating midpoint integrated linear finite elements, described in Section 3.1, as Crank–Nicolson discretization of an equivalent first-order system (Section 3.2). Exponential convergence is achieved by choosing the parameters by linking the Crank–Nicolson discretization with Padé approximants (Section 3.3). The result is a finite element mesh with complex element lengths, as discussed in Section 3.3. In Section 4 we show the applicability of the present method to general vector (elastic) equations. Section 5 contains numerical examples illustrating the exponential convergence and practical use of the proposed method.

2. Preliminaries

2.1. Problem statement

For the sake of focused discussion, consider the model problem in Fig. 1, where an elastic layer with three segments, each individually stratified in the vertical direction. Excitation is applied only at the vertical edges and interfaces, and the goal is to obtain the response at these locations. We consider the anti-plane shear deformation governed by the Helmholtz equation (Laplace equation being a special case when frequency $\omega = 0$):