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A hybrid reconstructed discontinuous Galerkin and continuous Galerkin finite element method for incompressible flows on unstructured grids

Aditya K. Pandare [1], Hong Luo [*,2]

North Carolina State University, Raleigh, NC, 27695, USA

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ABSTRACT

A hybrid reconstructed discontinuous Galerkin and continuous Galerkin method based on an incremental pressure projection formulation, termed $\mathrm{rDG}(\mathrm{P_nP_m}) + \mathrm{CG}(\mathrm{P_n})$ in this paper, is developed for solving the unsteady incompressible Navier–Stokes equations on unstructured grids. In this method, a reconstructed discontinuous Galerkin method ($\mathrm{rDG}(\mathrm{P_nP_m})$) is used to discretize the velocity and a standard continuous Galerkin method ($\mathrm{CG}(\mathrm{P_n})$) is used to approximate the pressure. The $\mathrm{rDG}(\mathrm{P_nP_m}) + \mathrm{CG}(\mathrm{P_n})$ method is designed to increase the accuracy of the hybrid $\mathrm{DG}(\mathrm{P_n}) + \mathrm{CG}(\mathrm{P_n})$ method and yet still satisfy Ladyženskaja–Babuška–Brezzi (LBB) condition, thus avoiding the pressure checkerboard instability. An upwind method is used to discretize the nonlinear convective fluxes in the momentum equations in order to suppress spurious oscillations in the velocity field. A number of incompressible flow problems for a variety of flow conditions are computed to numerically assess the spatial order of convergence of the $\mathrm{rDG}(\mathrm{P_nP_m}) + \mathrm{CG}(\mathrm{P_n})$ method. The numerical experiments indicate that both $\mathrm{rDG}(\mathrm{P_0P_1}) + \mathrm{CG}(\mathrm{P_1})$ and $\mathrm{rDG}(\mathrm{P_1P_2}) + \mathrm{CG}(\mathrm{P_1})$ methods can attain the designed 2nd order and 3rd order accuracy in space for the velocity respectively. Moreover, the 3rd order $\mathrm{rDG}(\mathrm{P_1P_2}) + \mathrm{CG}(\mathrm{P_1})$ method significantly outperforms its 2nd order $\mathrm{rDG}(\mathrm{P_0P_1}) + \mathrm{CG}(\mathrm{P_1})$ and $\mathrm{rDG}(\mathrm{P_1P_1}) + \mathrm{CG}(\mathrm{P_1})$ counterparts: being able to not only increase the accuracy of the velocity by one order but also improve the accuracy of the pressure.

1. Introduction

The discontinuous Galerkin methods [3,8–13,17,26,27,33] (DGM) have become popular for the solution of systems of conservation laws in computational fluid dynamics in the past few decades. The discontinuous Galerkin methods combine two advantageous features commonly associated with finite element and finite volume methods. As in classical finite element methods, the DGM achieve high order accuracy by means of high-order polynomial approximation within an element rather than by use of wider stencils as in the case of finite volume methods. The physics of wave propagation is, however, accounted for by solving Riemann problems that arise from the discontinuous representation of the solution at element

* Corresponding author.
E-mail address: hong_luo@ncsu.edu (H. Luo).

1 Graduate Research Assistant, Department of Mechanical and Aerospace Engineering.

2 Professor, Department of Mechanical and Aerospace Engineering.

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interfaces, which makes them similar to finite volume methods. The discontinuous Galerkin methods have many attractive features: (1) Their mathematical rigor implies useful mathematical properties with respect to conservation, stability and convergence; (2) The methods can be easily extended to higher-order (>2nd) approximations; (3) They are well suited for complex geometries since they can be applied on unstructured grids. In addition, the methods can also handle non-conforming elements, where the grids are allowed to have hanging nodes; (4) The methods are highly parallelizable, as they are compact and each element is independent; (5) Since the elements are discontinuous, and the inter-element communications are minimal, domain decomposition can be efficiently employed. The compactness also allows for structured and simplified implementation and coding; (6) They can easily handle adaptive strategies, since refining or coarsening a grid can be achieved without considering the continuity restriction commonly associated with the conforming elements; (7) The methods allow easy implementation of *hp*-refinement, for example, the order of accuracy, or shape, can vary from element to element. *p*-refinement can be achieved by simply increasing the order of the approximation polynomial. However, the DGM have their own weaknesses. In particular, compared to the finite element methods and finite volume methods, the DGM require solutions of systems of equations with more unknowns for the same grids. Consequently, these methods have been recognized as expensive in terms of both computational costs and storage requirements especially in the context of implicit methods, where the memory requirement for the Jacobian matrix grows quadratically with the order of the DG methods, thus leading to a significant increase in computational cost.

In order to reduce high costs of the DGM, a new family of reconstructed discontinuous Galerkin methods [11–13,23,27,28], termed PnPm schemes by Dumbser et al. and referred to as $rDG(P_nP_m)$ in this paper, have been developed for solving the compressible Euler and Navier–Stokes equations. In the $rDG(P_nP_m)$ methods, P_n indicates that a piecewise polynomial of degree of n is used to represent a DG solution, and P_m represents a reconstructed polynomial solution of degree of m $(m \geq n)$ that is used to compute the fluxes. The beauty of $rDG(P_nP_m)$ schemes is that they provide a unified formulation for both finite volume and DG methods, and contain both classical finite volume and standard DG methods as two special cases of $rDG(P_nP_m)$ schemes, and thus allow for a direct efficiency comparison. When $n = 0$, i.e. a piecewise constant polynomial is used to represent a numerical solution, $rDG(P_0P_m)$ is nothing but classical high order finite volume schemes, where a polynomial solution of degree m $(m \geq 1)$ is reconstructed from a piecewise constant solution. When $m = n$, the reconstruction reduces to the identity operator, and $rDG(P_nP_n)$ scheme yields a standard DG method. Obviously, the construction of an accurate and efficient reconstruction operator is crucial to the success of the $rDG(P_nP_m)$ schemes. In Dumbser's work, a higher order polynomial solution is reconstructed using a L2 projection, requiring it to be indistinguishable from the underlying DG solutions in the contributing cells in the weak sense. The resultant over-determined system is then solved using a least-squares method that guarantees exact conservation, not only of the cell averages but also of all higher order moments in the reconstructed cell itself, such as slopes and curvatures. However, this conservative least-squares reconstruction approach is computationally expensive, as the L2 projection, i.e., the operation of integration, is required to obtain the resulting over-determined system. Furthermore, the reconstruction might be problematic for a boundary cell, where the number of the face-neighboring cells might not be enough to provide the necessary information to recover a polynomial solution of a desired order. Fortunately, the projection-based reconstruction is not the only way to obtain a polynomial solution of higher order from the underlying discontinuous Galerkin solutions. In a reconstructed DG method using a Taylor basis [25,26] developed by Luo et al. for the solution of the compressible Euler and Navier–Stokes equations on arbitrary grids, a higher order polynomial solution is reconstructed by use of a strong interpolation, requiring point values and derivatives to be interpolated on the face-neighboring cells. The resulting over-determined linear system of equations is then solved in the least-squares sense. This reconstruction scheme only involves von Neumann neighborhood, and thus is compact, simple, robust, and flexible. Like the projection-based reconstruction, the strong reconstruction scheme guarantees exact conservation, not only of the cell averages but also of their slopes due to a judicious choice of the Taylor basis. A comparative study [27,29] on these reconstructed discontinuous Galerkin methods $rDG(P_1P_2)$ to solve the compressible Euler equations on arbitrary grids indicates that the rDG methods can deliver the desired 3rd order of accuracy. They also significantly improve the accuracy of the underlying 2nd order DG method. Thus, it can be concluded that the least-squares reconstruction method provides the best performance in terms of both accuracy and robustness.

A number of the DGM [1,2,4,20] have been developed for solving the incompressible Navier–Stokes equations. Bassi et al. [1,2] presented a DG method for the incompressible Navier–Stokes equations written in conservative form based on an artificial compressibility formulation. Botti, Di Pietro [4] and Kyriazis, Ekaterinaris [20] adopted the pressure-correction formulation [4–6,16,21,32] to solve the incompressible Navier–Stokes equations written in a non-conservative form. They use a continuous Galerkin $CG(P_n)$ and $CG(P_{n-1})$ discretizations for the pressure field and $DG(P_n)$ discretization for the velocity field in order to satisfy the Ladyženskaja–Babuška–Brezzi (LBB) condition. They report that the velocity converges with the order of $n+1$ and the pressure at a convergence rate of n for both $DG(P_n)+CG(P_n)$ and $DG(P_n)+CG(P_{n-1})$ spatial discretization. In addition, they observe a 2nd order of convergence in time for a range of Re $(10^2, 10^3, 10^4)$.

Based on the success of the rDG methods for the compressible flows, the objective of this work is to develop a rDG method for solving the incompressible Navier–Stokes equations based on a projection formulation. The resulting algorithm is expected to be high-order accurate in space; be able to handle both steady and unsteady problems. In spite of being high-order accurate, the algorithm must be computationally efficient and able to provide stable solutions to the incompressible Navier–Stokes equations. A reconstructed discontinuous Galerkin approximation $(rDG(P_nP_m))$ is used for the velocity field and a continuous Galerkin approximation $(CG(P_m))$ is used to discretize the pressure field. The developed hybrid method, $rDG(P_nP_m) + CG(P_m)$, inherently satisfies the so-called LBB condition and thus can effectively avoid the pressure checker-