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Anisotropic norm-oriented mesh adaptation for a Poisson problem

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A B S T R A C T

We present a novel formulation for the mesh adaptation of the approximation of a Partial Differential Equation (PDE). The discussion is restricted to a Poisson problem. The proposed norm-oriented formulation extends the goal-oriented formulation since it is equation-based and uses an adjoint. At the same time, the norm-oriented formulation somewhat supersedes the goal-oriented one since it is basically a solution-convergent method. Indeed, goal-oriented methods rely on the reduction of the error in evaluating a chosen scalar output with the consequence that, as mesh size is increased (more degrees of freedom), only this output is proven to tend to its continuous analog while the solution field itself may not converge. A remarkable quality of goal-oriented metric-based adaptation is the mathematical formulation of the mesh adaptation problem under the form of the optimization, in the well-identified set of metrics, of a well-defined functional. In the new proposed formulation, we amplify this advantage. We search, in the same well-identified set of metrics, the minimum of a norm of the approximation error. The norm is prescribed by the user and the method allows addressing the case of multi-objective adaptation like, for example in aerodynamics, adaptating the mesh for drag, lift and moment in one shot. In this work, we consider the basic linear finite-element approximation and restrict our study to L^2 norm in order to enjoy second-order convergence. Numerical examples for the Poisson problem are computed.

1. Introduction

This paper addresses anisotropic mesh adaptation. We focus on methods which build a somewhat optimal mesh defined by a parametrization using a Riemannian metric. A typical family of optimal metric-based methods for CFD is the family of Interpolation-based/Hessian-based methods. An attractive property of these methods is that they are based on a mathematical optimization principle.

Iso-distribution/equi-repartition Hessian-based methods tend to minimize a Sup or L^∞ norm of the (main term of) interpolation error with respect to a metric considered in a subset of metrics with a prescribed number of vertices. We refer to the two pioneering works [13,16] for the methods, the two pioneering works [1,31] for the analysis, and to [32,4]. The methods minimizing the L^p norm ($p < \infty$) of the interpolation error of one or several *sensors* depending on the CFD solution allow to better capture features of different scales in the solution. Cf. [34,22,14,23,5]. Sensors are solution-dependent fields chosen by the user according to their ability to take into account mesh-resolution difficulties of the flow to compute.

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The Hessian-based methods are particularly well adapted to the finite-element approximation of second-order elliptic PDEs. It is true, by the projection theorem, that a norm of the approximation error is bounded by the analog norm of the interpolation error but this concerns the H^1 norm while Hessian-based method concentrates on L^∞ or L^p norms. More generally, while taking into account the features of the PDE *solution*, these methods do not take into account the features of the *PDE* itself. This is penalizing in the case of systems, for which several sensors need to be chosen and weighted by the user. However, if sensors are wisely chosen, a good convergence of the whole approximate solution field to the exact solution field is usually observed.

Taking into account the influence of the PDE on the error through an *equation-based estimate* has also been an important research topic. The formulation of *goal-oriented methods* is an important step for a more justified error evaluation. It has been introduced in [7]. It relies on an *a posteriori* estimate. A good synthesis concerning *a posteriori* estimates is [36], see also [17]. An interest of *a posteriori estimate* is that it is expressed in terms of the approximate solution, assumed to be available in a mesh adaptation loop. A second interest is that it does not require the use of higher-order (approximate) derivatives, in contrast to truncation analyses. These estimates show accurately where the mesh should be refined. A method for deducing a better anisotropic mesh from an *a posteriori* estimate is proposed in [18], while a theory for H^p norms in [3] and a joint analysis of H^p and L^p norms of the error are presented in [2]. These methods cannot focus on an arbitrary user-specified error norm but rely on a particular one, specified by the variational formulation of the PDE. A more popular option is to choose, as accuracy target, a particular scalar output depending on the PDE solution. Any scalar output can be considered, except that difficulties can arise for the so-called non-admissible ones, according to [6]. An *a posteriori* estimate also allows for building *correctors* for goal-oriented analyses [19,30]. In [35], the goal-oriented approach is cleverly combined with the correction strategy of [30] and with the Hessian-based metric approach, still minimizing the interpolation error of a user-prescribed sensor.

A priori estimates generally rely on Taylor series, either through divided differences or through polynomial approximation of functions. Then, approximations of higher-order derivatives of solution need to be built from the approximate solution. This is a delicate job since there are not many proofs ensuring that a good approximation of a higher-order derivative of the exact solution can be recovered from the low-order approximate solution. However, since the development of the first recovery methods (see for example [38]), many numerical experiments tend to show that the method is useful and rather reliable.

A remarkable feature of the goal-oriented metric-based adaptation of [23,9] is the complete and coherent mathematical formulation of the mesh adaptation problem. Indeed, it takes the form of the optimization of a well-defined functional, namely the error for a prescribed scalar output, to be minimized with respect to a parameter, the metric, belonging to a well-identified and compact set. This strategy is applied to the discrete case in [37]. In [23,9], in order to analytically solve the optimum, an *a priori* analysis is developed. It restricts to the main asymptotic term of the local error in order to exhibit more easily the dependence with respect to metric.

Goal-oriented methods have strongly impacted the applications but, due to its formulation, a goal-oriented method has two inherent limitations. First, it does not naturally extend to several scalar outputs. This “multi-target” issue is well-known and a proposition for addressing it is made in [21]. Second, because they are specialized to a given scalar output, the features of the solution field which are not influencing this output might be neglected by the automatic mesh improvement. A goal-oriented method provides the convergence of the approximate prescribed scalar output to its continuous analog. But generally convergence does not hold for the whole flow field itself. To clarify this point, let us consider the mesh adaptative computation of a sonic boom footprint at the ground. The functional depends only on the pressure at the ground level. Now, many details of the flow on upper part of the aircraft do not influence the pressure at the ground level. This vanishing influence is taken into account by the adjoint state which also vanishes on these upper regions. Then, in these regions, the adapted mesh is not refined and the approximation of the flow field does not converge. See an illustration in [26]. As already noted, another limitation of a goal-oriented method is the scalar character of the error to reduce. It leads to use integrals of solution fields as in $(u - u_h, g)$, u being the exact solution, u_h its approximation and g a field prescribed by user. Now, these integrals are generally not sufficiently sensitive to oscillating deviations between u and u_h.

In the new norm-oriented formulation proposed in this paper, the user can prescribe a norm or a semi-norm $|u - u_h|$ of the error in order to minimize it with respect to the mesh. As a typical example of semi-norm, this can be the sum of square deviations on particular outputs. Let us take an example in aerodynamics. The semi-norm $|u - u_h| \equiv |C_l(u) - C_l(u_h)|^2 + |C_d(u) - C_d(u_h)|^2 + |C_m(u) - C_m(u_h)|^2$ will account for minimizing the errors on lift, drag and moment measured from flow solution u_h with respect to mesh. The proposed method will ultimately address this kind of semi-norm, assuming that, as for the goal-oriented method, the possible issue of a non-admissible norm according to [6] is solved. As for the goal-oriented method, the norm-oriented method takes into account the PDE features and, in the case where a *norm* is prescribed, it produces an approximate solution field which does converge to the exact one in this norm.

Although the proposed method is a rather general method extending to more complex CFD models, see for example [27], we consider in this paper a 2D Poisson problem discretized by the usual linear finite-element method. This choice is motivated first by the rather complete set of theoretical works available for the finite-element approximation of a Poisson problem. This amount of theoretical background reduces as much as possible (although far from completely) the heuristics to introduce in building the mesh adaptation analysis. In contrast, convergence results (in which functional spaces?) are not available for Euler equations, for example. A second motivation is the easy availability of exact solutions defined in a simple way. This allows to build a kind of benchmark allowing to compare mesh adaptation methods. Let us mention also that