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An efficient exponential time integration method for the numerical solution of the shallow water equations on the sphere

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ABSTRACT

The exponential propagation methods were applied in the past for accurate integration of the shallow water equations on the sphere. Despite obvious advantages related to the exact solution of the linear part of the system, their use for the solution of practical problems in geophysics has been limited because efficiency of the traditional algorithm for evaluating the exponential of Jacobian matrix is inadequate. In order to circumvent this limitation, we modify the existing scheme by using the Incomplete Orthogonalization Method instead of the Arnoldi iteration. We also propose a simple strategy to determine the initial size of the Krylov space using information from previous time instants. This strategy is ideally suited for the integration of fluid equations where the structure of the system Jacobian does not change rapidly between the subsequent time steps. A series of standard numerical tests performed with the shallow water model on a geodesic icosahedral grid shows that the new scheme achieves efficiency comparable to the semi-implicit methods. This fact, combined with the accuracy and the mass conservation of the exponential propagation scheme, makes the presented method a good candidate for solving many practical problems, including numerical weather prediction.

1. Introduction

The nonlinear partial differential equations, governing physical and chemical processes in continuous systems, are often solved using the method of lines [43] which leads to a large set of Ordinary Differential Equations (ODEs) of the form

$$\frac{du}{dt} = F(u,t), \qquad u(0) = u_0, \tag{1}$$

where $u \in \mathbf{R}^n$ is the state vector, n indicates the number of degrees of freedom, and F is the function describing all forcings; $F : \mathbf{R}^{n+1} \longrightarrow \mathbf{R}^n$.

The system described by Eq. (1) is typically stiff because it governs processes with different time scales. Consequently, the selection of an appropriate time integration scheme is of the foremost importance. In all situations when the stiffness comes from the linear part of F, the problem could be cast in a simpler form as

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$$\frac{du}{dt} = \mathcal{L}u + \mathcal{N}(u,t), \tag{2}$$

where $\mathcal{L}$ and $\mathcal{N}$ are linear and nonlinear part respectively.

There are abundant examples of partial differential equations which lead to a semi-discrete form of Eq. (2). They include (see Minchev [23] for review): Allen–Cahn, Burgers, Cahn–Hilliard, Kuramoto–Sivashinsky, Navier–Stokes, shallow water, Swift–Hohenberg, nonlinear Schrödinger, convection–diffusion and convection–reaction–diffusion equations. The numerical solution of the set of semi-linear system described by Eq. (2) is often obtained using semi-implicit schemes where the linear term $\mathcal{L}u$ is solved with the help of a method appropriate for the stiff problem (usually an implicit method) whereas the nonlinear part $\mathcal{N}(u,t)$ is solved explicitly. This methodology offers a rational compromise between requirements of accuracy and efficiency. The alternative strategy that has attracted increased attention in a number of diverse fields in recent years, is the exponential time integration method. It requires the evaluation of functions related to the exponential of the Jacobian matrix [14]. Methods belonging to this class offer very high accuracy and stability without a severe time step restriction of the explicit numerical schemes. Since their introduction in the late 1950s, several numerical software packages providing various implementations have been proposed [14]. For a recent review, see [17] and the references therein.

The exponential integration schemes based on the static splitting given by Eq. (2) were discussed by Beylkin et al. [4] and Cox and Matthews [9]. The basic idea of these methods is quite simple. We begin from the multiplication of Eq. (2) by factor $e^{\mathcal{L}t}$ and the integration over time step Δt to obtain

$$u_{n+1} = e^{\mathcal{L}\Delta t}u_n + \int_0^{\Delta t} e^{\mathcal{L}(\Delta t-\tau)}\mathcal{N}(u(n\Delta t+\tau))d\tau, \tag{3}$$

where u_n, u_{n+1} are solutions at times n and $n+1$, Δt is the time step and τ is the time. The exponential term is then evaluated using the methods discussed in [36] and the integral of the nonlinear term is approximated with an appropriate quadrature. It was shown that the methods of this class offer a very good accuracy and a realistic representation of the high frequencies in contrast to the traditional semi-implicit schemes [9].

Considering the results obtained with exponential integration methods in various areas of science, it is justified to investigate them in the context of numerical weather prediction where the selection of a time integration scheme is a key element. Traditionally, the meteorological equations are solved using the well-established Semi-Implicit Semi-Lagrangian (SISL) integration schemes, first introduced in the atmospheric community by André Robert [32] and [33]. The main advantage of these methods is that they are not limited by a stability-based CFL condition. Hence the time step size can be chosen solely on the basis of a desired accuracy.

The efficiency and robustness of the SISL methods for integrating meteorological equations led to their common use in many of the meteorological centers. Semi-Lagrangian schemes were further advanced by the development of the accurate parallel algebraic solvers [6,27]. The fully conservative algorithms were also implemented. Most of these advances have been motivated by the need to use parallel computing architectures in the optimum manner. The same consideration is the driving force behind the search of alternative techniques.

Different versions of the exponential time integrators have been studied in the meteorological context by numerous authors. Archibald et al. [2] used the scheme of Beylkin et al. [4], based on the assumption of static splitting described by Eq. (2), to solve the shallow water equations on the cubed sphere. Clancy and Pudykiewicz [8] applied the exponential propagation methods based on the dynamic linearization [40] to the shallow water system on an icosahedral geodesic grid. The basic principle of this method is outlined briefly as follows. After expanding Eq. (1) in a Taylor series around state $u(t_n)$ at t_n we obtain

$$\frac{du}{dt}(t) = F_n + \mathcal{J}_n \cdot (u(t) - u_n) + R(u(t)), \tag{4}$$

where $u_n = u(t_n)$, $F_n = F(u_n)$, $\mathcal{J}_n = \frac{dF}{du}(u_n)$ and

$$R(u(t)) = F(u(t)) - F_n - \mathcal{J}_n \cdot (u(t) - u_n). \tag{5}$$

The use of integrating factor $e^{-\mathcal{J}_n t}$ on Eq. (4) yields

$$\frac{d}{dt}(e^{-\mathcal{J}_n t}u(t)) = e^{-\mathcal{J}_n t}(F_n - \mathcal{J}_n u_n) + e^{-\mathcal{J}_n t}R(u(t)). \tag{6}$$

Integrating over $[t_n, t_n + \Delta t_n]$ and multiplying by $e^{\mathcal{J}_n(t_n+\Delta t_n)}$ leads to the integral form

$$u(t_n + \Delta t_n) = u_n + (e^{\mathcal{J}_n \Delta t_n} - I)\mathcal{J}_n^{-1}F_n + \int_{t_n}^{t_n+\Delta t_n} e^{\mathcal{J}_n(t_n+\Delta t_n - t)}R(u(t))dt, \tag{7}$$

where Δt_n indicates Δt at the time step number n and I is the identity matrix. The methodology of linearization applied in this paper is discussed in [15,16,18].