Journal of Computational Physics 321 (2016) 105–125

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

A comparison of high-order polynomial and wave-based methods for Helmholtz problems

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ARTICLE INFO

Article history:
Received 14 December 2015
Received in revised form 20 May 2016
Accepted 21 May 2016
Available online xxxx

Keywords:
Comparison
High-order finite element method
Wave-based discontinuous Galerkin method
Helmholtz problems

ABSTRACT

The application of computational modelling to wave propagation problems is hindered by the dispersion error introduced by the discretisation. Two common strategies to address this issue are to use high-order polynomial shape functions (e.g. *hp*-FEM), or to use physics-based, or Trefftz, methods where the shape functions are local solutions of the problem (typically plane waves). Both strategies have been actively developed over the past decades and both have demonstrated their benefits compared to conventional finite-element methods, but they have yet to be compared. In this paper a high-order polynomial method (*p*-FEM with Lobatto polynomials) and the wave-based discontinuous Galerkin method are compared for two-dimensional Helmholtz problems. A number of different benchmark problems are used to perform a detailed and systematic assessment of the relative merits of these two methods in terms of interpolation properties, performance and conditioning. It is generally assumed that a wave-based method naturally provides better accuracy compared to polynomial methods since the plane waves or Bessel functions used in these methods are exact solutions of the Helmholtz equation. Results indicate that this expectation does not necessarily translate into a clear benefit, and that the differences in performance, accuracy and conditioning are more nuanced than generally assumed. The high-order polynomial method can in fact deliver comparable, and in some cases superior, performance compared to the wave-based DGM. In addition to benchmarking the intrinsic computational performance of these methods, a number of practical issues associated with realistic applications are also discussed.

1. Introduction

Numerical dispersion is the main source of inaccuracy in the conventional finite element method when solving Helmholtz problems at high frequencies. For instance with linear elements the numerical error scales like k^2h with wavenumber k and elements of size h. As k increases, the number of elements per wavelength must also increase to maintain the same level of accuracy, which rapidly becomes prohibitively expensive. A number of different strategies have been explored to address this issue [1,2]. The present paper provides a detailed comparison of two of the most common techniques: high-order polynomial methods and wave-based methods.

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http://dx.doi.org/10.1016/j.jcp.2016.05.045
0021-9991/

High-order finite element methods replace the standard, low-order Lagrange polynomials with higher-order functions providing superior interpolation properties. Different families of polynomials have been considered such as Bernstein, Hermite or Lobatto polynomials (comparisons of these can be found in [3,4]). Fourier series or Tchebychev polynomials are also used with spectral methods [5]. In this work, the framework of p-FEM is used to construct a continuous, high-order approximation with Lobatto shape functions for solutions of the Helmholtz equation [6,7]. With this method the dispersion error is drastically reduced [8] and it has been shown to be a valid approach to address the pollution effect and tackle large-scale problems [9,10]. The p-FEM approach provides exponential convergence when increasing the polynomial order p. It is also well-suited for p-adaptivity where the polynomial order is adjusted locally in the computational domain [11]. Finally, the hierarchic nature of the Lobatto shape functions leads to efficient algorithms for solving the same model over a range of frequencies [10].

Wave-based methods represent another category of methods developed to remedy the pollution problem (a detailed review of these methods can be found in [12]). The rationale is to incorporate *a priori* knowledge about the local behaviour of the solution into the numerical method. This is generally achieved by using local canonical solutions of the governing equations to build an approximation basis and these methods are therefore also called Trefftz methods. For the Helmholtz equation, the local solutions are generally plane waves, but other alternatives can be used, such as Bessel functions [13]. The central argument for using a basis of plane waves is that it follows directly from the governing equations and thus it allows embedding key features of the underlying physics into the numerical method. It is generally assumed that a plane-wave basis naturally provides better accuracy compared to polynomials since the latter bears no relation with the governing equations. One of the objectives of this paper is to assess this assumption.

A number of wave-based methods rely on discontinuous formulations where the solution is approximated with plane waves in each element and the continuity between elements is imposed weakly. This includes the Ultra Weak Variational Formulation (UWVF) [14–16], the least square method [17], and the wave-based discontinuous Galerkin method [18]. They differ in the way the inter-element continuity is formulated but it is now recognised that these methods are all variants of the same discontinuous Galerkin method using different numerical fluxes to ensure continuity between elements [19]. The DGM framework is particularly well suited for adaptivity since it is straightforward to change the number of plane waves in each element. A number of theoretical analyses have also been reported for this particular method [20–23]. Results from [24] indicate that the wave-based DGM exhibits exponential convergence with respect to the number of degrees of freedom for very general element shapes, including in the presence of strong mesh refinements. This paves the way to for the development of fully automatic hp-adaptive versions of the method and a posteriori error estimators are being investigated [25]. In this work, the wave-based DGM from [18] is used. Numerical results in [19,26] indicate that it is more efficient than the UWVF and least-square methods.

The Discontinuous Enrichment Method (DEM) is another example of methods using discontinuous interpolations [27]. Enrichment functions are added to the set of conventional shape functions to construct the approximate solution in each element. Lagrange multipliers defined on the edges between elements enforce the continuity of the solution. This method has been used to solve the Helmholtz equation [28] and extended to other fields such as fluid-structure interactions [29]. Removing the conventional shape functions from the DEM leads to a wave-based discontinuous Galerkin method with Lagrange multipliers [28]. The DEM leads to a significant reduction of the number of DOFs when compared to linear, quadratic and quartic Lagrange finite elements for a fixed accuracy in 2D [28,30] and 3D [31].

Other Trefftz methods relying on discontinuous interpolations include the wave-based method [32] and the variational theory of complex rays (VTCR) [33], but these two methods cannot be formulated in the DG framework [12]. Results in [34] indicate that the UWVF tends to outperform the VTCR for simple Helmholtz problems.

The Partition of Unity FEM (PUFEM) constructs an approximation of the solution by multiplying enrichments functions with the standard FEM shape functions [35]. The benefit of this approach is that the resulting approximation space is naturally continuous. The downside is the calculation of the element matrices, which can be costly because it involves highly-oscillatory integrals with products of exponentials and polynomials. Elements of comparisons of PUFEM with the UWVF can be found in [36,37] and the UWVF is relatively better conditioned and generally more efficient.

Both high-order polynomial FEM and wave-based methods have seen sustained developments over the past decades and have been applied to a wide range of disciplines (acoustics, aero-acoustics, linear elasticity, water waves, electromagnetism, porous materials). While partial elements of comparison exist in the literature [28,30,31], there hasn't been a systematic and detailed assessment of their relative merits for Helmholtz problems. This is the focus of the present paper. Using a number of two-dimensional benchmark problems it provides a detailed comparison of the accuracy, cost and conditioning of two of these methods: p-FEM and the wave-based discontinuous Galerkin method.

The structure of the paper is as follows. Section 2 describes the two numerical methods and the benchmark problems are presented in Section 3. Section 4 discusses the different metrics used to assess the cost and accuracy of the numerical methods. The interpolation properties of each basis (polynomial and wave-based) are compared in Section 5. This is followed in Section 6 by the convergence and relative performance of the two methods to achieve a given accuracy. Practical aspects of the two computational methods are then discussed in Section 7.