Journal of Computational Physics 321 (2016) 126–150

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

Discontinuous approximation of viscous two-phase flow in heterogeneous porous media

Raimund Bürger [a,c,*], Sarvesh Kumar [b], Sudarshan Kumar Kenettinkara [c], Ricardo Ruiz-Baier [d]

[a] Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile
[b] Department of Mathematics, Indian Institute of Space Science and Technology, Thiruvananthapuram 695 547, Kerala, India
[c] CI²MA, Universidad de Concepción, Casilla 160-C, Concepción, Chile
[d] Mathematical Institute, Oxford University, Andrew Wiles Building, Radcliffe Observatory Quarter, Woodstock Road, OX2 6GG Oxford, UK

ARTICLE INFO

Article history:
Received 2 October 2015
Received in revised form 12 April 2016
Accepted 20 May 2016
Available online 27 May 2016

Keywords:
Two phase flow
Brinkman equations
Runge–Kutta discontinuous Galerkin methods
Stabilization
Finite volume element methods
Discontinuous fluxes

ABSTRACT

Runge–Kutta Discontinuous Galerkin (RKDG) and Discontinuous Finite Volume Element (DFVE) methods are applied to a coupled flow-transport problem describing the immiscible displacement of a viscous incompressible fluid in a non-homogeneous porous medium. The model problem consists of nonlinear pressure–velocity equations (assuming Brinkman flow) coupled to a nonlinear hyperbolic equation governing the mass balance (saturation equation). The mass conservation properties inherent to finite volume-based methods motivate a DFVE scheme for the approximation of the Brinkman flow in combination with a RKDG method for the spatio-temporal discretization of the saturation equation. The stability of the uncoupled schemes for the flow and for the saturation equations is analyzed, and several numerical experiments illustrate the robustness of the numerical method.

1. Introduction

1.1. Scope

It is the purpose of this contribution to introduce a new numerical approach for the accurate simulation of viscous two-phase flow (for instance of oil and water) in a heterogeneous porous medium. The flow of the mixture is governed by the Brinkman model while the interaction of the two phases can be described by the fractional flow formalism, which translates into a transport equation of one phase that involves a nonlinear flux function. Since the medium is considered non-homogeneous, not only do the flow properties undergo abrupt changes (as does the permeability of the medium), but also the flux characterization will exhibit discontinuities associated to different nonlinearities adjacent to a discontinuity in the medium.

Specifically, the governing model is defined as follows. Let us consider a mixture of two fluids, a wetting phase and a non-wetting phase (e.g., water and oil) identified by the indices w and n, with saturations $\phi_{\mathrm{w}} = \phi$ and $\phi_{\mathrm{n}} = 1 - \phi$,

* Corresponding author.
E-mail addresses: rburger@ing-mat.udec.cl (R. Bürger), sarvesh@iist.ac.in (S. Kumar), skumar@ci2ma.udec.cl (K. Sudarshan Kumar), ruizbaier@maths.ox.ac.uk (R. Ruiz-Baier).

http://dx.doi.org/10.1016/j.jcp.2016.05.043
0021-9991/

respectively, in a domain $\Omega \subset \mathbb{R}^2$. If the fluids are incompressible and capillary forces are negligible, then the following model is adequate to describe the viscous motion of the mixture in a porous medium:

$$\partial_t \phi + \operatorname{div} \mathbf{F}(\phi, \boldsymbol{u}, \boldsymbol{x}) = 0 \qquad \text{in } \Omega \times (0, T), \tag{1.1a}$$

$$\mathbf{K}^{-1}(\boldsymbol{x})\boldsymbol{u} - \mathbf{div}\big(\mu(\phi)\boldsymbol{\varepsilon}(\boldsymbol{u}) - p\mathbf{I}\big) - \phi\boldsymbol{g} = \mathbf{0} \qquad \text{in } \Omega \times (0, T), \tag{1.1b}$$

$$\operatorname{div} \boldsymbol{u} = 0 \qquad \text{in } \Omega \times (0, T), \tag{1.1c}$$

supplied with suitable initial and boundary conditions. Here and elsewhere, div and **div** are used for the standard divergence of an arbitrary vector $\boldsymbol{v} \in \mathbb{R}^2$ and matrix $\mathbf{A} \in \mathbb{R}^{2\times 2}$, i.e.,

$$\operatorname{div} \boldsymbol{v} := \sum_{i=1}^{2} \frac{\partial v_i}{\partial x_i}, \qquad \mathbf{div}\,\mathbf{A} := \left(\sum_{j=1}^{2} \frac{\partial a_{ij}}{\partial x_j} \right)_{1 \le i \le 2}.$$

The primal unknowns are the volume average flow velocity of the mixture $\boldsymbol{u}$, the saturation ϕ, and the pressure field p. In addition, $\mu(\phi)\boldsymbol{\varepsilon}(\boldsymbol{u}) - p\mathbf{I}$ is the Cauchy stress tensor, $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\mathrm{T}})$ is the infinitesimal rate of strain, $\mu = \mu(\phi)$ is the saturation-dependent viscosity,

$$\mathbf{F}(\phi, \boldsymbol{u}, \boldsymbol{x}) = f(\phi)\boldsymbol{u} + b(\phi)\mathbf{K}(\boldsymbol{x})\boldsymbol{g} \tag{1.2}$$

is a nonlinear flux vector, $\mathbf{K}$ is the permeability tensor of the medium, and $\boldsymbol{g}$ is the gravity acceleration. Moreover, f and b are given fractional flow functions that will be specified later.

We assume that $\mathbf{K}$ is symmetric, uniformly bounded and positive definite, and that μ and $\mathbf{K}^{-1}$ satisfy

$$\mu, \mu' \in \operatorname{Lip}(\mathbb{R}_+); \quad \exists \gamma_1, \mu_{\min}, \mu_{\max} > 0 : \forall s \in \mathbb{R}_+ : \ \mu_{\min} < \mu(s) < \mu_{\max}, \ |\mu'(s)| \le \gamma_1, \tag{1.3}$$

$$\forall \boldsymbol{x} \in \Omega; \quad \exists\, k_1, k_2 > 0 \text{ such that } 0 < k_1 \le \mathbf{K}^{-1}(x) \le k_2, \tag{1.4}$$

where the last inequalities are understood in a component-wise sense.

The numerical approximation of the system (1.1) calls for advanced techniques capable to accurately capture flux discontinuities, and robust flow solvers that satisfy discrete maximum principles (ϕ must remain bounded between zero and one), produce divergence-free approximations of velocity, and ensure local mass conservation. It is known that classical methods (e.g. pure upwind finite volumes or primal finite elements and others) do not capture the behavior of the flow near the interface and may yield nonphysical solutions unless some sort of monotonicity-preserving slope limiter or high-order reconstruction is added. Other possible remedies include multi-point flux approximation, high-order DG or other nonconforming methods, phase field models, XFEM, or level set strategies. Regarding the flow equations (in this case, of Brinkman type), we are interested in accurate methods that would permit a natural development of error estimates (therefore associated with finite element formulations), that are mass conservative by construction (therefore related to mixed formulations, or to pure finite volume schemes), and which could easily handle unstructured meshes. These aspects are the prime motivation for proposing Discontinuous Finite Volume Element (DFVE) methods for the approximation of the Brinkman equations (1.1b), (1.1c) and Runge–Kutta Discontinuous Galerkin (RKDG) approximation of the saturation equation (1.1a). The main novelty of this paper is the particular numerical treatment of the two-phase flow equations (1.1) through the implementation of the RKDG scheme together with the DFVE method and the corresponding stability analysis. Since we have included the effect of gravity and the medium is heterogeneous, the flux function in (1.1a) is non-monotone in ϕ and discontinuous in $\boldsymbol{x}$. These properties require an appropriate choice of the numerical flux in the RKDG formulation. Showing the robustness of numerical results that arise from incorporating the so-called DFLU flux of [2] into the RKDG formulation further adds to the novelty.

1.2. Related work

The Brinkman model exhibits the well-known advantage that one can represent both Stokes and Darcy flows without imposing explicit interface (e.g. Beavers–Joseph–Saffman) conditions. The latter are not necessarily consistent with mixture theory, and are quite difficult to treat numerically. In fact, several vectorial and scalar Lagrange multipliers need to be incorporated to impose energy conservation, as is done, for instance, in [3,16,28].

The model (1.1) is similar to the continuum-based description of sedimentation and consolidation of suspended particles recently discretized by FVE-related methods [14,41]. In turn, multiscale FVE methods were applied in [22,25] for the simulation of two-phase flow in porous media. An adaptive FVE was proposed in [36] for a steady convection–diffusion–reaction problem. In [12,30] the authors propose and analyze unified DFVE methods for Stokes-transport and Darcy-transport problems, respectively; in [14] an axisymmetric sedimentation problem is discretized with a combination of continuous FVE and DFVE for Stokes and a degenerate parabolic equation, continuous FVE-based formulations for coupled Darcy and transport were proven to satisfy discrete maximum principles [23] and applied to Navier–Stokes–transport couplings in [37,41]; and a hybrid mixed FE–DFVE has been recently introduced in [40] for a larger class of multiphase flows in rigid and compliant porous media.