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Short note

A first-order hyperbolic system approach for dispersion

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1. Introduction

Dispersion effects play a fundamental role in many applications involving hydrodynamics. At the large scale, the flow is dominated by advection, while the dissipative effects are more important at the microscopic level. At the mesoscopic level (the intermediate level) the dispersive effects become important, as it is the case, for example, in non-linear optics, electromagnetism, quantum mechanics [1–3], relativity and Bose–Einstein condensates [4,5], atmospheric, coastal, and fluvial hydrodynamics [6–10], magma, highly viscous fluids and/or capillary effects [11–13]. The main physical effects associated with dispersion are the appearance of dispersive (or undulating) shocks, and the existence of smooth traveling solitary waves, which may produce complex interactions with one another. These systems, independent of the physical nature of the involved medium, admit a mesoscale hydrodynamic model, which consists of a set of Partial Differential Equations (PDEs). These PDEs are mostly regularizations of hyperbolic models that, in one-dimension, may be written as

$$\partial_t u + \partial_x F(u) = \mathcal{D}, \tag{1}$$

where, in a classical sense (endowed with an entropy pair, with a diagonalizable flux Jacobian $F'(u)$, etc.), the left hand side defines a hyperbolic model. Depending on the application and the physical hypotheses made, the regularization on the right hand side may take different forms, and have a dissipative and/or dispersive character (see e.g., [14–18,12]). Therefore, term $\mathcal{D}$, may be a combination of the following forms:

(a) dissipative regularization $\mathcal{D} = \nu\, \partial_{xx} u$,
(b) dispersive regularization with time derivative $\mathcal{D} = \epsilon\, \partial_{xxt} u$,
(c) fully dispersive regularization $\mathcal{D} = \epsilon\, \partial_{xxx} u$.

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The form (a) is a classical viscous regularization, while form (b) can be recast as a first-order evolutionary PDE with an embedded steady state second-order time independent elliptic problem; i.e.,

$$\partial_t w = -\partial_x f\ , \qquad -\partial_{xx} u + u = w.$$

We previously showed e.g., in Refs. [19–21] how one can construct a very accurate numerical approximations of problems of form (a) and the elliptic form of (b), by first reformulating the system of PDEs as a first-order hyperbolic system. The schemes proposed with the hyperbolic formulation of PDE systems are upwind and highly accurate for both the solution u and its gradient u_x, and have a natural potential for extension on arbitrary unstructured meshes as illustrated in Refs. [22,23].

The presence of a third-order derivative term in form (c), however, introduces discretization difficulties, which are often related to the understanding of the type of stencil that is required to approximate these high-order derivative terms, the stability of the method used, and the imposition of boundary conditions. In Refs. [24,16], possible solutions to some of these issues are proposed, where the authors showed very good results with a non-hyperbolic first-order system reformulation of a PDE and careful discretization of the fluxes. The hyperbolic reformulation of dispersive PDEs similar to the one initially proposed for diffusion in Ref. [19] alleviate the above mentioned issues. However, it is shown and proved in Ref. [25] that the hyperbolic formulation of a dispersive PDE in the form given in Ref. [19] is not possible. Thus, we are motivated to introduce an alternative hyperbolic formulation that is carefully designed for general dispersive PDEs that are relevant to, for example, quantum mechanics, relativistic hydrodynamics, and coastal engineering applications, such as the Korteweg–de Vries (KdV) equation [14,17,24].

In this work, we first present a first-order system for a pure dispersion equation, and show how the proposed formulation can be made hyperbolic. As an intermediate generalization, we then show that an advective–dispersive PDE (such as the classical KdV) can also be made fully-hyperbolic as well. The fully-hyperbolic advection–dispersion system could be useful in imposing characteristics boundary condition. A practical extension of the proposed hyperbolic dispersion system for general advective–diffusive–dispersive PDEs follows next. We then present some numerical examples by applying the high-order residual-distribution (RD) scheme of Ref. [21] to the proposed system, and solving general dispersive PDEs, including the classical KdV equation, on randomly distributed nodes. We verify the order of accuracy of the scheme for both the solution, the gradient, and the Hessian (second derivative) with the use of method of manufactured solution, and show that the RD scheme applied to the proposed hyperbolic system produces accurate solution, gradient and Hessian with equal order of accuracy. The ability to obtain accurate gradient and Hessian are very important as they are used in many hydrodynamic dispersive models to define physically relevant quantities (e.g., potentials, energy, flow properties at an arbitrary depth, etc.). We also present solutions for the zero dispersion limit of conservation laws and demonstrate that the RD scheme applied to the proposed hyperbolic system is robust and can capture physical oscillations associated with the generated dispersive shocks.

2. Hyperbolic dispersion

In this section, we start with a time-dependent dispersive PDE, and reformulate it to a first-order system than can be successfully transformed to a first-order hyperbolic system.

Consider the following linear dispersive PDE that is often referred to as the Airy equation,

$$\partial_t u = \epsilon\, \partial_{xxx} u, \tag{2}$$

where ϵ is the dispersion coefficient (positive or negative). Following the process we outlined in Ref. [20] (although other choices may also be possible), we consider here the semi-discrete form of Eq. (2) obtained with some implicit time integration scheme:

$$\frac{\alpha}{\Delta t} u = \epsilon\, \partial_{xxx} u + s(x),$$

where α and $s(x)$ depend on the time discretization and the known values of u [20,21]. We then replace the semi-discrete PDE by the steady limit of the pseudo-time dependent system

$$\begin{aligned}
\partial_\tau u &= \epsilon\, \partial_x q - \frac{\alpha}{\Delta t} u + s(x),\\
\partial_\tau p &= \frac{1}{T_\epsilon}\left(\partial_x u - p - \gamma u\right),\\
\partial_\tau q &= \frac{1}{T_\epsilon}\left(\gamma\, \partial_x u + \partial_x p - q\right),
\end{aligned} \tag{3}$$

where τ is the pseudo time, t is the physical time, and T_ϵ and γ are, respectively, the dispersion relaxation time and an arbitrary constant, both to be defined later.

It is easy to verify that u in the above system satisfies the original semi-discrete dispersion equation at pseudo steady state (see also Refs. [20,21]). Note that, the proposed system with $\gamma = 0$ reduces to the first-order system formulation that is