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ColDICE: A parallel Vlasov–Poisson solver using moving adaptive simplicial tessellation

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ABSTRACT

Resolving numerically Vlasov–Poisson equations for initially cold systems can be reduced to following the evolution of a three-dimensional sheet evolving in six-dimensional phase-space. We describe a public parallel numerical algorithm consisting in representing the phase-space sheet with a conforming, self-adaptive simplicial tessellation of which the vertices follow the Lagrangian equations of motion. The algorithm is implemented both in six- and four-dimensional phase-space. Refinement of the tessellation mesh is performed using the bisection method and a local representation of the phase-space sheet at second order relying on additional tracers created when needed at runtime. In order to preserve in the best way the Hamiltonian nature of the system, refinement is anisotropic and constrained by measurements of local Poincaré invariants. Resolution of Poisson equation is performed using the fast Fourier method on a regular rectangular grid, similarly to particle in cells codes. To compute the density projected onto this grid, the intersection of the tessellation and the grid is calculated using the method of Franklin and Kankanhalli [65–67] generalised to linear order. As preliminary tests of the code, we study in four dimensional phase-space the evolution of an initially small patch in a chaotic potential and the cosmological collapse of a fluctuation composed of two sinusoidal waves. We also perform a "warm" dark matter simulation in six-dimensional phase-space that we use to check the parallel scaling of the code.

1. Introduction

Stars in galaxies and dark matter in the Universe can be described as a smooth self-gravitating collisionless fluid following Vlasov–Poisson equations,

$$\frac{\partial f}{\partial t} + \mathbf{u}.\nabla_{\mathbf{r}} f - \nabla_{\mathbf{r}}\phi.\nabla_{\mathbf{u}} f = 0, \tag{1}$$

$$\Delta_{\mathbf{r}}\phi = 4\pi G\rho = 4\pi G \int f(\mathbf{r},\mathbf{u},t)\, \mathrm{d}^3\mathbf{u}, \tag{2}$$

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where $f(\mathbf{r}, \mathbf{u}, t)$ represents the phase-space density at position $\mathbf{r}$, velocity $\mathbf{u}$ and time t, ϕ is the gravitational potential and G is the gravitational constant.

In this article, we focus on the cold case, relevant to the dynamics of cold dark matter. In the concordant model of large scale structure formation [123,124], the matter content in Universe is indeed dynamically dominated by a cold and collisionless component, designated by “dark” matter as it does not emit detectable light or radiation. The cold nature of this component implies that the phase-space distribution function is initially concentrated on a phase-space sheet: at the macroscopic level, the thickness of this sheet is virtually null:

$$f(\mathbf{r}, \mathbf{u}, t = t_{\rm i}) = \rho_{\rm i}(\mathbf{r})\, \delta_{\rm D}[\mathbf{u} - \mathbf{u}_{\rm i}(\mathbf{r})], \tag{3}$$

where $\rho_{\rm i}(\mathbf{r})$ is the initial density distribution, $\mathbf{u}_{\rm i}$ the initial velocity field and $\delta_{\rm D}$ the Dirac distribution function. In this case, the matter is initially concentrated on a $D = 3$ hypersurface in $2D = 6$-dimensional phase-space.

Liouville’s theorem states that the phase-space density is conserved along trajectories,

$$f[\mathbf{r}(t), \mathbf{u}(t), t] = \text{constant}, \tag{4}$$

for any point $[\mathbf{r}(t), \mathbf{u}(t)]$ following the equations of motion. This means that topological properties of the phase-space distribution function are conserved during motion, in particular that the phase-space sheet, i.e. the region where f is non-null remains a non-self-intersecting three-dimensional hypersurface at all times.

In our understanding of large scale structure formation, the initial density field $\rho_{\rm i}(\mathbf{r})$ is close to constant and the initial velocity field, when subtracted from the expansion of the Universe, is very small: the large scales structures observed today, such as clusters of galaxies, filaments and underdense regions nearly empty of galaxies [see, e.g., [72]], grew from small initial fluctuations in the density field through gravitational instability [121].

Vlasov–Poisson equations are traditionally resolved numerically with a N-body approach [see, e.g., [20,43,55,52], for reviews]. There exist many kinds of N-body codes, among those one can list direct summation codes (PP for particle–particle interactions) [3,1], particle–mesh (PM) codes [56,109,111,98,29] coming initially from plasma physics [85], treecodes [10,16, 80,31,139] as well as hybrid codes such as P^3M (PM combined with PP for local interactions) [85,60], treePM (PM combined with treecode for local interactions) [151,14,137], adaptive mesh refinement codes (AMR) [148,141,135,70,102,99,143,36] and AP^3M (P^3M with AMR) [47]. In all these methods, which mainly differ from each other by the way Poisson equation is solved, the phase-space distribution function is represented by an ensemble of particles, that is a set of Dirac functions in phase-space interacting with each other through gravitational forces. To avoid numerical instabilities due to close collisions, the gravitational force is smoothed at scales smaller than a softening parameter ε which corresponds to the local grid resolution in mesh based methods such as PM and AMR.

The representation of a smooth distribution with a discrete set of macro-particles can have non-trivial consequences on the numerical behaviour of the system.[1] Close N-body encounters and collective effects due to the shot noise of the particles may drive the system away from the expected solution in the mean field limit [2,81,71,136,26,28,25,54,91]. For instance, shot noise of the particles can introduce significant distortions of the phase-space sheet as well as nonlinear resonant instabilities [7,46,44], which can have dramatic consequences on the numerical behaviour of the system, particularly in the cold case [113,112,9]. Hence, the fine structure of dark matter halos is still the object of debates despite multiple convergence studies with N-body simulations [118,119,114,89,90,125,138,140,33,97].

For all these reasons and because computational power now allows it, it is justified to explore alternative numerical routes and to try solving Vlasov–Poisson directly in phase-space without resorting to particles. This is important to confirm many results obtained with the traditional N-body approach and that are used to test the cold dark matter scenario paradigm against observations.

In the warm case, i.e. where the initial velocity dispersion is non-negligible, there exists a very rich literature about Vlasov solvers, mainly coming from plasma physics. Many of these solvers exploit directly Liouville theorem (4). Among them, one can cite the famous splitting scheme of Cheng and Knorr [41] first applied to astrophysical systems by [68,150, 120,69]. In this algorithm, the phase-space distribution function is sampled on a mesh. It is updated between two time steps by following backwards the motion of test particles and by using an interpolation scheme to compute f from the particles positions at previous time step. The procedure is performed in a split fashion, by decomposing the Hamiltonian motion into a “drift” (e.g., position update using velocity) and a “kick” (e.g., velocity update using acceleration). This algorithm is *semi-Lagrangian*, in the sense that it relies on the calculation of characteristics.

Many other grid based Vlasov solvers have been proposed since the seminal contribution of Cheng and Knorr, most of them being of semi-Lagrangian nature [see, e.g., [133,132,153,145,117,134,63,11,23,7,144,50,49,126,40,130,79,74], but this list is far from being exhaustive]. For instance one can mention the recent Vlasov–Poisson simulations of Yoshikawa, Yoshida & Umemura [152] in six-dimensional phase-space using the positive flux conservation scheme [63].

While it can be of interest for describing warm astrophysical systems such as galaxies, or cosmological fluids with significant velocity dispersion such as the neutrino distribution in the Universe, sampling the phase-space distribution function

[1] We do not discuss here the self-consistent field method [42,146,82,86], because it is seldomly used. In this approach, the projected density and the gravitational potential are represented on a finite set of carefully chosen smooth functions of which the respective weights are computed from a set of particles following the equations of motion. There is no softening needed in this method and the noise introduced by the tracers is different from what is expected in standard N-body simulations but is still unquestionably present [81].