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Time-domain implementation of an impedance boundary condition with boundary layer correction

E.J. Brambley [a,*], G. Gabard [b]

[a] Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Wilberforce Road, Cambridge, CB3 0WA, United Kingdom
[b] Institute of Sound and Vibration Research, University of Southampton, Highfield, Southampton, SO17 1BJ, United Kingdom

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ABSTRACT

A time-domain boundary condition is derived that accounts for the acoustic impedance of a thin boundary layer over an impedance boundary, based on the asymptotic frequency-domain boundary condition of Brambley (2011) [25]. A finite-difference reference implementation of this condition is presented and carefully validated against both an analytic solution and a discrete dispersion analysis for a simple test case. The discrete dispersion analysis enables the distinction between real physical instabilities and artificial numerical instabilities. The cause of the latter is suggested to be a combination of the real physical instabilities present and the aliasing and artificial zero group velocity of finite-difference schemes. It is suggested that these are general properties of any numerical discretization of an unstable system. Existing numerical filters are found to be inadequate to remove these artificial instabilities as they have a too wide pass band. The properties of numerical filters required to address this issue are discussed and a number of selective filters are presented that may prove useful in general. These filters are capable of removing only the artificial numerical instabilities, allowing the reference implementation to correctly reproduce the stability properties of the analytic solution.

1. Introduction

Since acoustic liners are routinely used within aeroengines to reduce noise, it is imperative that computational aeroacoustics (CAA) simulations include models of acoustic liners. In the frequency domain, acoustic liners are easily modelled as an impedance surface, where an oscillatory fluid pressure $\mathrm{Re}\big(p' \exp\{\mathrm{i}\omega t\}\big)$ at the surface gives rise to a normal fluid velocity $\mathrm{Re}\big(v_s \exp\{\mathrm{i}\omega t\}\big)$ through the surface, linked through the complex impedance $Z(\omega) = p'/v_s$. The entire physical modelling of the acoustic lining is encapsulated within the impedance $Z(\omega)$, for which numerous empirical and physical models exist [e.g. 2–5]; for further details, see Ref. [6] and the references therein.

Typically in aeroacoustics, oscillations in the fluid are small perturbations to a steady mean flow, so that the total velocity is $\boldsymbol{u}_0(\boldsymbol{x}) + \boldsymbol{u}'(\boldsymbol{x}, t)$ and similarly for the pressure and density. The presence of mean flow complicates the application of an impedance boundary condition. For example, rather than setting $\boldsymbol{u}' \cdot \boldsymbol{n} = p'/Z$ at the boundary, where $\boldsymbol{n}$ is the normal to the acoustic lining, the boundary condition widely used is

* Corresponding author. Tel.: +44 1223 760457; fax: +44 1223 765900.
E-mail addresses: E.J.Brambley@damtp.cam.ac.uk (E.J. Brambley), gabard@southampton.ac.uk (G. Gabard).

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$$i\omega \boldsymbol{u}' \cdot \boldsymbol{n} = \big(i\omega + \boldsymbol{u}_0 \cdot \nabla - (\boldsymbol{n} \cdot \nabla \boldsymbol{u}_0) \cdot \boldsymbol{n}\big) p'/Z. \tag{1}$$

This equation is known as the Myers, or Ingard–Myers, boundary condition, after Ingard [7] and Myers [8], and corresponds to matching normal acoustic displacement between the fluid and the acoustic liner rather than normal velocity. For flat surfaces where $(\boldsymbol{n} \cdot \nabla \boldsymbol{u}_0) \cdot \boldsymbol{n} \equiv 0$, equation (1) was shown by Eversman and Beckemeyer [9] and Tester [10] to be the correct limit of an infinitely thin inviscid boundary layer.

For CAA simulations the mean flow is assumed known (for example, from prior RANS calculations), and the small perturbations are to be calculated. Numerical schemes for calculating these small perturbations have different requirements from those used for calculating the steady flow in order to ensure low dispersion and dissipation, and many optimized schemes exist [e.g. 11–13]. Requiring low dissipation means that instabilities are often found at under-resolved scales, of the order of half the Nyquist frequency (i.e. four points per wavelength spatially), necessitating selective filtering [14,13,15]. In time domain simulations, this selective filtering often takes the form of a weak low-pass spatial filter applied at every point at every time step.

For certain impedance models $Z(\omega)$, including the mass–spring–damper impedance [4,16] for which $Z = R + im\omega - iK/\omega$ and the Extended Helmholtz Resonator impedance [5,17] for which $Z = R + im\omega - i\beta\cot(\omega L - i\varepsilon/2)$, time-domain versions of the frequency-domain boundary condition (1) are possible. However, high-frequency numerical instabilities are invariably present in time-domain simulations [4,17,18] when such impedances are used with slipping mean flow using the Myers boundary condition (1). These numerical instabilities are different from the instabilities due to the use of low-dissipation numerical schemes mentioned above [19]. Stabilizing the boundary condition requires the use of strong, wide-band filters to indiscriminately remove any form of instability. This is not surprising, since the underlying mathematical model being simulated is illposed [20] and supports arbitrarily quick exponential growth at arbitrarily short wavelengths. One way to regularize this problem is to consider a thin but non-zero thickness inviscid boundary layer over the lining. Since resolving this boundary layer, while possible, is computationally more expensive [e.g. 21] and introduces its own stability difficulties in the form of a continuous spectrum [22–24], it is convenient instead to modify the Myers boundary condition to include the effects of a thin boundary layer [25,26]. Not only does this regularize the problem, but Ref. [25] has been shown to provide significantly better accuracy than the Myers boundary condition [27]. These regularizations have until now been restricted to the frequency-domain, and it is one purpose of this paper to present a time-domain finite-difference implementation of the modified boundary condition of Ref. [25].

It should be noted that flow past a non-rigid boundary is often unstable, both in theory [28,29] and in practice [30,31], as can be appreciated by considering the flapping of a flag in the wind [32]. While the modified boundary conditions mentioned above remove the illposedness of arbitrarily fast exponential growth caused by overly simple modelling assumptions, they should therefore still be expected to result in most cases in a convectively unstable system [26,33]. One other aim of this paper is therefore to describe general numerical difficulties that arise when simulating unstable linear systems. In particular, careful distinction is needed between genuine instabilities of the system being simulated and artificial instabilities introduced by the numerical discretization [19]. While there have been previous investigations of special cases, such as the careful numerical treatment of the continuous spectrum by Marx [24], here we propose for general simulations a general class of artificial instabilities caused by a combination of instability of the underlying physical system and the finite resolution of the numerical discretization. These artificial instabilities are of a similar nature to those caused directly by under-resolution and aliasing [e.g. 14], but are distinct on two accounts: firstly, they can be prevented by filtering only at the impedance boundary rather than throughout the fluid; and secondly, under-resolution in space leads to well-resolved exponential growth in time, rather than leading to dispersive and under-resolved temporal behaviour.

While we are concerned here with only inviscid flows, the techniques described here will be equally applicable to modified boundary conditions that incorporate viscosity [34–36], provided the modified boundary conditions remain well-posed. It is worth noting that viscosity by itself does not regularize the illposedness due to the assumption of an infinitely thin boundary layer [35], but viscosity is likely to be important for accuracy in certain situations [37,38], and for stabilizing the well-posed inviscid instabilities [36].

In order to verify that the numerical scheme developed here accurately reproduces the correct results and stability of the underlying equations, we will here consider a simple situation of a time-harmonic line source in a uniform flow past an acoustic liner which admits an analytic solution [39]. Following a description of the general method in §2, this simple situation is described in §3, together with the numerical scheme used and a comparison of numerical results with the analytic solution. A discussion of the important subtleties of this numerical scheme owing to the convective instability of the underlying equations is given in §4, in light of which the design and optimization of several boundary condition filters is described in §5. The conclusion in §6 also discusses possible future extensions of this work, including its application to other numerical schemes (such as finite elements) and the inclusion of viscosity.

2. Mathematical formulation

We consider a fluid with velocity $\boldsymbol{u}$, pressure p and density ρ. Neglecting viscosity, the governing equations are the Euler equations, given in Appendix A. We write the time-independent mean flow with a subscript zero and the small time-dependent perturbation with a prime, so that $\rho' = \rho - \rho_0$ is the small time-dependent density perturbation and $(\rho\boldsymbol{u})' = \rho\boldsymbol{u} - \rho_0\boldsymbol{u}_0$ is the small time-dependent momentum perturbation. The small perturbations are therefore governed by the Linearized Euler Equations (LEE), also given in Appendix A.