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Approximation of skewed interfaces with tensor-based model reduction procedures: Application to the reduced basis hierarchical model reduction approach

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ABSTRACT

In this article we introduce a procedure, which allows to recover the potentially very good approximation properties of tensor-based model reduction procedures for the solution of partial differential equations in the presence of interfaces or strong gradients in the solution which are skewed with respect to the coordinate axes. The two key ideas are the location of the interface either by solving a lower-dimensional partial differential equation or by using data functions and the subsequent removal of the interface of the solution by choosing the determined interface as the lifting function of the Dirichlet boundary conditions. We demonstrate in numerical experiments for linear elliptic equations and the reduced basis-hierarchical model reduction approach that the proposed procedure locates the interface well and yields a significantly improved convergence behavior even in the case when we only consider an approximation of the interface.

1. Introduction

Fluid flow problems such as subsurface flow or blood flow problems often feature one distinguished (dominant) direction along which the essential dynamics develop. Therefore, tensor-based model reduction procedures such as the proper generalized decomposition (PGD) method, the hierarchical model reduction (HMR), and the reduced basis-hierarchical model reduction (RB-HMR) approach are well suited to compute an efficient and accurate approximation of the full(-dimensional) solution of the underlying partial differential equation (PDE). The common idea of such tensor-based model reduction procedures is to approximate the full solution by a truncated tensor product decomposition of the form $p_m(x, y) = \sum_{l=1}^{m} \bar{p}_l(x)\phi_l(y)$, where x, y lie in the computational domain Ω and are associated with different coordinate axes. The resulting model reduction approaches then differ from one another in the way the tensor products $\bar{p}_l(x)\phi_l(y)$, $l = 1, \ldots, m$ are computed.

In the PGD method, introduced in [1,20], the tensor products are determined by iteratively solving the Euler–Lagrange equations associated with the considered problem. Alternatively, and in some cases equivalently, they may be computed as the minimizer of the variational functional corresponding to the considered PDE [19,5]. For an overview on the PGD method we refer to [6,7].

In contrast, the HMR approach, introduced in [31–33] and studied in a more general geometric setting in [12,26], considers a reduced space which is a combination of the full (Finite Element) solution space along the dominant (flow) direction

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with a reduction space spanned by orthonormal basis functions $\{\phi_l\}_{l=1}^{m}$ in the so-called transverse direction. The function p_m then solves a reduced problem obtained by a Galerkin projection onto the reduced space. While in [31–33,12,26] the reduction space is chosen a priori as the span of trigonometric or Legendre polynomials, a highly nonlinear approximation is employed for the construction in the RB-HMR approach [22,23,29,28]. To this end, first a parametrized problem in the transverse direction is derived from the full dimensional problem, where the parameters reflect the influence from the unknown solution in the dominant direction. Then, reduced basis (RB) techniques [24,27] are applied for the efficient construction of the reduction space from snapshots of the parametrized transverse problem, exploiting their good approximation properties [10,17]. Thus, both in the construction of the solution manifold of the parametrized lower-dimensional problem and in the subsequent choice of the basis functions, information on the full solution is included to obtain a fast convergence of the reduced solution to the full one. In general, this yields an improved convergence rate compared to a priori chosen reduction spaces [23,28].

In spite of their mentioned good performance for say fluid flow problems, the approximation capacity of tensor-based model reduction procedures suffers considerably if the target solution exhibits an interface, i.e. a steep gradient or even a discontinuity, which is skewed with respect to the coordinate axes. Such behavior can often be encountered in fluid flow problems and particularly in subsurface flow, where, depending on the permeability of the soil, the saturation profile may form a skewed interface along the water table. This deteriorated convergence behavior is due to the fact that for a full approximation of the skewed interface the saturation or concentration profile in each point x in the dominant direction has to be included. In this article we introduce a new ansatz to tackle this problem. We propose to first approximately locate the interface by solving a lower-dimensional model or for simple model problems to infer the location of the interface from data functions. We assume that we have Dirichlet data available at the positions where the interface intersects the boundary of the considered computational domain. Thus we can then infer an approximation of the shape of the interface from the known Dirichlet boundary conditions. Otherwise an approximate shape of the interface can be computed in a preprocessing step. Finally, we prescribe the obtained saturation or concentration profile as the lifting function of the Dirichlet boundary conditions. In this way, we hope to remove the part of the full solution, which causes the bad convergence rate from the approximation process and therefore significantly improve the convergence behavior of the employed tensor-based model reduction approach. This will be demonstrated in numerical experiments.

Alternative to our approach, in [13] an interface or shock propagating in time is included in a time-dependent basis, which is spanned by the eigenfunctions of a linear Schrödinger operator and yields a numerical approximation of a Lax pair. In [18] a reduced basis in space is constructed via a proper orthogonal decomposition of snapshots and the evolution of the coefficients in time is computed by a suitable mapping and thus in an equation-free manner. In the case of parametrized PDEs it is well-known that convection dominated evolution equations where shocks may develop are difficult to tackle with RB methods [27,24] if linear spaces are employed. The reason for this is that similar to the setting of the skewed interface considered in this article the solution for nearly every time step has to be included in the basis, which deteriorates the approximation properties of the RB space. Therefore, in [21] a nonlinear approximation is applied by employing the method of freezing to decompose the target solution into a shape and group component. Then RB methods are applied to approximate the former while the group component say captures a drift of the interface. Also in [30] the authors propose to employ a nonlinear approximation strategy for the approximation of the solution of parametrized conservation laws in one space dimension. The approach in [30] consists of a partition of the domain induced from a suitable approximation of the shock curve such that the solution in each obtained subdomain is regular. The empirical interpolation method [2] — an interpolation strategy from the RB framework — is used to reconstruct the smooth parts of the solution in the subdomains.

The remainder of this article is organized as follows. In Section 2 we first describe our approach for the location of the interface using the example of subsurface flow and subsequently outline how the location of the interface can be inferred from data functions for linear advection–diffusion problems (Section 2.1). Afterwards, we demonstrate for linear advection–diffusion problems how the information on the location of the interface can be used to remove the interface from the model reduction procedure in Section 2.2. In Section 3 we exemplify this ansatz for the RB-HMR method and present an approach for the derivation of a lower-dimensional parametrized problem particularly suited for the presence of interfaces, which will be validated in Section 4. The capacity of the ansatz proposed in Section 2 to improve the convergence behavior is demonstrated in Section 4 for linear problems for the RB-HMR approach in several numerical experiments, including a test case, where we do not include the exact interface but only an approximation.

2. An ansatz for approximating skewed interfaces with tensor-based model reduction approaches

Let $\Omega \subset \mathbb{R}^2$ denote the computational domain with Lipschitz boundary $\partial\Omega$, $\Sigma_D \subset \partial\Omega$ the Dirichlet boundary, and $\Sigma_N \subset \partial\Omega$ the Neumann boundary. We require that Σ_D has positive Hausdorff measure. We assume that Ω can be considered as a two-dimensional fiber bundle:

$$\Omega = \bigcup_{x \in \Omega_{1D}} \{x\} \times \omega_x,$$

where $\Omega_{1D} = (x_0, x_1)$ and ω_x denotes the transverse fiber associated with $x \in \Omega_{1D}$. Note that the generalization to domains with a more complex geometry is straightforward [25]. We define for any $x \in \Omega_{1D}$ the mapping $\psi(\cdot; x) : \omega_x \to \hat{\omega}$ between the fiber ω_x associated with $x \in \Omega_{1D}$ and a reference fiber $\hat{\omega}$ with $\hat{\omega} =]y_0, y_1[$. Furthermore, we introduce the mapping