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Spectral approximation methods and error estimates for Caputo fractional derivative with applications to initial-value problems

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ABSTRACT

In this paper, we revisit two spectral approximations, including truncated approximation and interpolation for Caputo fractional derivative. The two approaches have been studied to approximate Riemann–Liouville (R–L) fractional derivative by Chen et al. and Zayernouri et al. respectively in their most recent work. For truncated approximation the reconsideration partly arises from the difference between fractional derivative in R–L sense and Caputo sense: Caputo fractional derivative requires higher regularity of the unknown than R–L version. Another reason for the reconsideration is that we distinguish the differential order of the unknown with the index of Jacobi polynomials, which is not presented in the previous work. Also we provide a way to choose the index when facing multi-order problems. By using generalized Hardy's inequality, the gap between the weighted Sobolev space involving Caputo fractional derivative and the classical weighted space is bridged, then the optimal projection error is derived in the non-uniformly Jacobi-weighted Sobolev space and the maximum absolute error is presented as well. For the interpolation, analysis of interpolation error was not given in their work. In this paper we build the interpolation error in non-uniformly Jacobi-weighted Sobolev space by constructing fractional inverse inequality. With combining collocation method, the approximation technique is applied to solve fractional initial-value problems (FIVPs). Numerical examples are also provided to illustrate the effectiveness of this algorithm.

1. Introduction

The history of fractional calculus is almost as long as the integer one, which goes back to the Leibniz's note in his list to L'Hospital, dated 30 September 1695. After a long exploration of many mathematicians (including Euler, Laplace, Fourier, etc., see [1] for the history), present-day definitions of fractional order are given. Compared to R–L version, fractional derivative in Caputo sense is more likely to be used in physical problems for its advantage in dealing with initial condition [2]. In fact the Caputo fractional derivative is an integro-differential operator, defined by a convolution of a weak singular kernel and classical derivative of a given function, which can be used to model non-Markovian behavior of spatial or temporal processes. So fractional calculus has attracted considerable interest of researchers in different fields during past decades. Furthermore,

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models established by fractional differential equations (FDEs) have been successfully applied to several branches of science, such as physics, chemistry and engineering (see e.g. [3,4] and the references therein).

Numerical methods for solving FDEs have been attracting more and more interest in recent years. The key of these methods is how to approximate the fractional derivative. In [5], fractional centered difference was defined and was proved it could represent for Riesz fractional derivative. Furthermore, in [6] this scheme was proved to be of second order accuracy for Riesz fractional derivative. Numerical algorithms were established in [7] to calculate the fractional integral and the Caputo derivative. Li et al. [8] developed numerical method based on the piecewise polynomial interpolation to approximate the fractional integral and the Caputo derivative, and to solve FDEs. An automatic quadrature approach based on the Chebyshev polynomials was proposed in [9] for approximating the Caputo derivative. In [10], Li derived recursive formulas to compute fractional order integrals and fractional order derivatives of the Legendre, Chebyshev and Jacobi polynomials. And with using collocation method, multi-term problems could be solved. In [11], a point interpolation method (PIM) was applied to approximate R–L fractional derivative, after which space fractional diffusion equation was solved numerically. Besides, some other methods such as the L1, L2 and L2C schemes, are utilized frequently to discretize fractional order derivative in the procedure of solving FDEs (e.g. [12–16]).

Up to now, finite difference is still the main approach to approximate fractional derivatives. However, most of these schemes' convergence order is no more than 2. Kumar and Agrawal [17] proposed a block-by-block method for a class of FIVPs. Later Huang et al. [18] demonstrated that the convergence order of this method is at least 3. A higher-order difference scheme was proposed by Cao and Xu in [19], which was proved to be of $(3+\alpha)$th order for $\alpha \le 1$ and of 4th order for $\alpha > 1$.

Spectral methods have been applied to approximate fractional derivatives. However, for the difficulty in computing the fractional derivative of general Jacobi polynomials with indexes $\gamma, \delta \neq 0$ analytically, most researchers prefer the Legendre polynomials. In fact, the following two formulas are always used with Galerkin method to solve fractional diffusion or fractional reaction–diffusion equation (e.g. in [20,21]),

$$ {}_{-1}^{L}D_x^{\mu} L_n(x) = \frac{\Gamma(n+1)}{\Gamma(n-\mu+1)}(1+x)^{-\mu} J_n^{(\mu,-\mu)}(x), \quad x \in [-1,1] \tag{1.1} $$

$$ {}_{x}^{L}D_1^{\mu} L_n(x) = \frac{\Gamma(n+1)}{\Gamma(n-\mu+1)}(1-x)^{-\mu} J_n^{(-\mu,\mu)}(x), \quad x \in [-1,1] \tag{1.2} $$

where $L_n(x)$ and $J_n^{(\mu,-\mu)}(x)$ are the Legendre polynomial and Jacobi polynomial of nth degree respectively, ${}_{-1}^{L}D_x^{\mu}$ and ${}_{x}^{L}D_1^{\mu}$ are the left and right R–L fractional derivatives respectively.

For general Jacobi polynomials, thanks to Bateman fractional integral formula, the following formulas hold

$$ {}_{-1}^{L}D_x^{\mu}\left((1+x)^{\delta} J_n^{(\gamma,\delta)}(x)\right) = (1+x)^{\delta-\mu}\frac{\Gamma(n+\delta+1)}{\Gamma(n+\delta-\mu+1)} J_n^{(\gamma+\mu,\delta-\mu)}(x) \quad (\gamma \in \mathbb{R}, \delta > -1) \tag{1.3} $$

$$ {}_{x}^{L}D_1^{\mu}\left((1-x)^{\gamma} J_n^{(\gamma,\delta)}(x)\right) = (1-x)^{\gamma-\mu}\frac{\Gamma(n+\gamma+1)}{\Gamma(n+\gamma-\mu+1)} J_n^{(\gamma-\mu,\delta+\mu)}(x) \quad (\gamma > -1, \delta \in \mathbb{R}) \tag{1.4} $$

For further details, we refer interested readers to [33]. Zayernouri and Karniadakis studied the function $(1-x)^{\delta} J_n^{(\gamma,\delta)}(x)$ $(n = 0, 1, \cdots)$ in [31] and pointed out it is a family of eigenfunctions of a fractional Sturm–Liouville operator. Most recently Chen et al. [33] referred to this kind of functions as generalized Jacobi functions (GJFs) and elaborated its properties. Furthermore, spectral approximation results for these GJFs were derived in the non-uniformly Jacobi-weighted space involving fractional derivatives (we let $\mathfrak{B}_{\gamma,-\delta}^{p}(I)$ denote the space) and Petrov–Galerkin spectral method was constructed for a class of prototypical FDEs with convergence analysis.

In fact, one key property the authors utilized in [33] to obtain the estimate results in $\mathfrak{B}_{\gamma,-\delta}^{p}(I)$ is: for R–L fractional derivative operator, it holds

$$ {}_{-1}^{L}D_x^{\delta+l} = D^{l}\,{}_{-1}^{L}D_x^{\delta}, \qquad {}_{x}^{L}D_1^{\delta+l} = (-D)^{l}\,{}_{x}^{L}D_1^{\delta}. $$

However, this property is no longer valid for fractional derivative in Caputo sense. In this paper, we will provide two ways to approximate Caputo derivative: truncated approximation and interpolation. The basis we use are with the form $(x-a)^{\delta}\tilde{J}_n^{(\gamma,\delta)}$, where $\tilde{J}_n^{(\gamma,\delta)}$ is the shifted Jacobi polynomial defined on $[a,b]$. The main theme of this paper is to discuss the approximation results for $\delta = m \in \mathbb{N}^{+}$ in the classical Sobolev space $B_{\gamma,-m}^{p}(I)$ (also referred to as non-uniformly Jacobi-weighted space) and give the optimal estimate. What makes it different from [33] is that the estimate holds for any $\alpha \le m$. By establishing fractional inverse inequality, the optimal estimate for interpolation is also derived. In fact, by a modified technique, it is demonstrated that the convergence speed of the two approaches in L^2 is not less than $O(N^{\alpha-p})$ as well. Furthermore we will also prove the truncated approximation can also lead to spectral convergence with respect to the norm $\|\cdot\|_{\infty}$ for smooth data.

Corresponding to the results for $\delta = m$, we also present the estimate of truncated approximation for $\delta \in (m-1, m)$, which is actually a generalization of [33]. In addition, we will give the error estimate for the fractional interpolation proposed in [32] at the end of Section 3.