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Stabilization of numerical interchange in spectral-element magnetohydrodynamics

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ABSTRACT

Auxiliary numerical projections of the divergence of flow velocity and vorticity parallel to magnetic field are developed and tested for the purpose of suppressing unphysical interchange instability in magnetohydrodynamic simulations. The numerical instability arises with equal-order C^0 finite- and spectral-element expansions of the flow velocity, magnetic field, and pressure and is sensitive to behavior at the limit of resolution. The auxiliary projections are motivated by physical field-line bending, and coercive responses to the projections are added to the flow-velocity equation. Their incomplete expansions are limited to the highest-order orthogonal polynomial in at least one coordinate of the spectral elements. Cylindrical eigenmode computations show that the projections induce convergence from the stable side with first-order ideal-MHD equations during h-refinement and p-refinement. Hyperbolic and parabolic projections and responses are compared, together with different methods for avoiding magnetic divergence error. The projections are also shown to be effective in linear and nonlinear time-dependent computations with the NIMROD code Sovinec et al. [17], provided that the projections introduce numerical dissipation.

1. Introduction

Magnetohydrodynamic (MHD) computations of magnetically confined plasma must address the numerical realization of localized interchange. In the absence of resistive dissipation, i.e. ideal MHD, the threshold for linear instability lies at a finite amount of "bad" magnetic curvature [1,2]. Bending of magnetic field-lines provides a restoring force, but it vanishes for resonant helical perturbations that align wavefronts with magnetic field (**B**) over entire surfaces of closed field-lines. Mathematically, the model is singular at these surfaces, and numerical representation of the resonant effect influences convergence properties [3–6]. The challenge lies in the fact that responses depend on numerical behavior at the limit of spatial resolution, where convergent numerical methods usually have their greatest level of truncation error. Convergence on local interchange "from the unstable side" or "from above" means having too large a growth rate at finite resolution of a physical instability or producing numerical interchange for physically stable conditions. Lütjens and Luciani aptly recognized this as a problematic source of numerical noise in nonlinear simulations of long timescale dynamics [6]. This article describes a practical approach to addressing numerical interchange in simulations that use finite- and spectral-element representations of C^0 continuity across element borders. It is conceptually related to spectral filtering of incompressible fluid simulations

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[7], to a penalty method for MHD eigenvalue computations [4], and to finite/spectral element stabilization techniques [for example, 8–11].

Our interest is nonlinear time-dependent simulation of nonideal MHD for magnetic confinement. This model admits viscosity that can stabilize localized interchange and facilitate computations. However, dissipation of dynamics perpendicular to magnetic field is weak in high-temperature plasma, and adding enough conventional viscosity to stabilize numerical interchange can distort dynamics. In principle, it is possible to address numerical interchange with sufficiently fine meshing for physical levels of viscosity. However, resonances that are susceptible to interchange can exist across the entire region of closed magnetic flux, and computing with globally fine resolution is computationally challenging and inefficient. The many resonances that occur in a region of bad magnetic curvature would also foil attempts to stabilize selected modes with spatially localized viscous dissipation.

The numerical representation of interchange was first studied in the context of ideal-MHD eigenvalue computation for static equilibria. The fundamental dependent fields are the components of the displacement vector, and perturbations in pressure and magnetic field are eliminated analytically prior to discretization. The resulting force operator is a second-order differential operator in space, and it is self-adjoint [12]. Finite-element methods (FEM) for this problem benefit from distinct expansions for the flux-normal, parallel, and "cross" components of the displacement vector, and integration by parts leaves only first-order derivatives of basis and test functions. To conform with analytical variational methods, only the radial component in cylindrical computations needs C^0 continuity in the radial direction [3,13]. The other components may be discontinuous, and the degree of polynomials may be chosen to represent incompressible displacements without truncation error within elements. This allows the cylindrical computations to distinguish shear, parallel compression, and perpendicular compression at the scale of the mesh, thereby avoiding numerical spectral pollution. Like other applications of FEM, the conforming representation for low-order elements tends to be too stiff, and weakly growing internal modes in shaped cross sections are easily missed with slow convergence from the stable side [14]. Hybrid methods include auxiliary expansions for element-averaged dependent fields to improve the representation of bending and to achieve convergence from the unstable side [3]. Degtyarev and Medvedev analyzed the hybrid methods and refined them by introducing a numerical penalty term with a resolution-dependent coefficient that enhances bending-like energy at the finest scale of a mesh [4].

The early work on ideal-MHD eigenvalue computation provides a basis for time-dependent computation with weak dissipation, but the latter requires separate expansions for flow velocity, magnetic field, pressure (or temperature), and particle or mass density. When solving equations for the primitive physical fields, ideal effects, including the stabilizing bending response, are represented through first-order spatial derivatives in temporally first-order equations. Although the equation for the displacement vector in eigenvalue computation and the ideal part of time-dependent equations represent the same physical system, they are mathematically distinct and require different numerical approaches. The one taken by Lütjens and Luciani expands magnetic flux and the streamfunction for flow with polynomials of different degree [6]. Results described in Sect. 4 concur with their finding that this method achieves convergence on interchange from the stable side. However, it allows other numerical modes when using primitive-field expansions in our spectral-element computations.

The approach presented here takes advantage of the spectral-element representation, which allows bases of arbitrary polynomial degree [15,16]. We project flow divergence and parallel vorticity onto incomplete, discontinuous polynomial expansions that include only the highest-order orthogonal polynomial in at least one of the element coordinates. Mathematically coercive responses to these projections are added to the evolution equations for components of flow velocity. This improves the numerical stability of the MHD computations, similar to other stabilization methods for FEM. Like spectral filtering [7], the auxiliary responses act on the shortest scales of the representation. In practice their effect on resolved scales is negligible.

The following section describes the equations and local interchange. Section 3 presents the new projection-based approach for stabilizing numerical interchange. Section 4 shows results from numerical eigenvalue computations for numerical systems with and without the new projections. Linear and nonlinear time-dependent results are presented in Sect. 5, followed by discussion and conclusions in Sect. 6.

2. Equations and local interchange

The nonlinear non-ideal MHD system is the model of interest for applications to magnetized plasma, but their linearized form in the ideal limit is useful for investigating numerical interchange behavior. Thus, both are presented in the first part of this section. The second part describes the gravitational instability in a sheared-slab configuration as a relatively simple realization of MHD interchange. It highlights the importance of responses to parallel vorticity and compression, motivating the numerical approaches that are introduced in Sect. 3.

2.1. Nonlinear and linear MHD equations

The developments described in this paper have been applied to the NIMROD (Non-Ideal MHD with Rotation) code [17], which solves time-dependent nonlinear and linear problems of extended MHD. In normalized units, the most basic nonlinear single-fluid system solved by NIMROD is

$$\rho\left(\frac{\partial}{\partial t}+\mathbf{V}\cdot\nabla\right)\mathbf{V}=\mathbf{J}\times\mathbf{B}-\nabla P-\nabla\cdot\underline{\Pi}, \tag{1}$$