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Reduced basis ANOVA methods for partial differential equations with high-dimensional random inputs

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ABSTRACT

In this paper we present a reduced basis ANOVA approach for partial deferential equations (PDEs) with random inputs. The ANOVA method combined with stochastic collocation methods provides model reduction in high-dimensional parameter space through decomposing high-dimensional inputs into unions of low-dimensional inputs. In this work, to further reduce the computational cost, we investigate spatial low-rank structures in the ANOVA-collocation method, and develop efficient spatial model reduction techniques using hierarchically generated reduced bases. We present a general mathematical framework of the methodology, validate its accuracy and demonstrate its efficiency with numerical experiments.

1. Introduction

Over the past few decades there has been a rapid development in numerical methods for solving partial differential equations (PDEs) with random inputs. This explosion in interest has been driven by the need of conducting uncertainty quantification for practical problems. In particular, uncertainty quantification for problems with high-dimensional random inputs gains a lot of interest. High-dimensional inputs exist in many practical problems, for example, problems with inputs described by random processes with short correlation lengths. This paper is devoted to high-dimensional uncertainty quantification problems.

To the authors' knowledge, there exist two main kinds of computational challenges for efficiently solving these high-dimensional uncertainty quantification problems in the context of PDEs: curse of dimensionality for the parameter space, and large-rank structures in spatial approximations. The curse of dimensionality is an obstacle to apply stochastic spectral methods [1–5]. As discussed in our earlier study [6], high-dimensional random inputs can also lead to large spatial ranks, which make it difficult to apply model reduction techniques for spatial approximations.

Many new methods are developed to resolve these challenging high-dimensional and large-rank problems. For parameter space discretization, ANOVA methods [7–15] are developed to decompose a high-dimensional parameter space into a union of low-dimensional spaces, such that stochastic collocation methods can then be efficiently applied. Besides ANOVA, adaptive sparse grids [16,3,17–20], multi-element collocation [21] and compressive sensing methods [22–24] are also developed to discretize high-dimensional parameter spaces. For efficient spatial approximation, localized reduced basis methods are developed to resolve large-rank problems, for example, model reduction based on spitting parameter domains [25,26]

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and that based on spatial domain decomposition methods [27–29]. In addition, efficient decomposition methods for both parameter and spatial spaces are developed in [30–32], and general distributed uncertainty quantification approaches are proposed in [33–36].

In this paper we focus on the ANOVA decomposition method. We note that low-dimensional parameter spaces generated in the ANOVA decomposition [12,14] can also lead to low-rank structures in spatial approximations. To capture these low-rank spatial structures, we develop a hierarchical reduced basis method. Since these low-rank structures give very small sizes of reduced bases, our proposed method can significantly improve the computational efficiency of the ANOVA method. In addition, we remark that model reduction methods to enhance the performance of stochastic spectral methods are also investigated in [37,6,38,39].

An outline of the paper is as follows. We present our problem setting and review the ANOVA-collocation combination in the next section. In Section 3, we review the reduced basis methods for parameterized PDEs. Our main algorithm is presented in Section 4. Numerical results are discussed in Section 5. Second 6 concludes the paper.

2. Problem setting and ANOVA decomposition

Let $D \subset \mathbb{R}^d$ $(d = 2, 3)$ denote a spatial domain which is bounded, connected and with a polygonal boundary ∂D, and $x \in \mathbb{R}^d$ denote a spatial variable. Let ξ be a vector which collects a finite number of random variables. The dimension of ξ is denoted by M, i.e., we write $\xi = [\xi_1, \ldots, \xi_M]^T$. The probability density function of ξ is denoted by $\pi(\xi)$. In this paper, we restrict our attention to the situation that ξ has a bounded and connected support. We next assume the support of ξ to be I^M where $I := [-1, 1]$, since any bounded connected domain in $\mathbb{R}^M$ can be mapped to I^M. The physics of problems considered in this paper are governed by a PDE over the spatial domain D and boundary conditions on the boundary ∂D. The *global* problem solves the governing equations which are stated as: find $u(x, \xi) : D \times I^M \to \mathbb{R}$, such that

$$\mathcal{L}(x, \xi; u(x, \xi)) = f(x) \qquad \forall (x, \xi) \in D \times I^M, \tag{1}$$

$$\mathfrak{b}(x, \xi; u(x, \xi)) = g(x) \qquad \forall (x, \xi) \in \partial D \times I^M, \tag{2}$$

where $\mathcal{L}$ is a partial differential operator and $\mathfrak{b}$ is a boundary operator, both of which can have random coefficients. f is the source function and g specifies the boundary conditions. In the rest of this section, we review the ANOVA decomposition [20,14] and stochastic collocation methods [3].

2.1. ANOVA decomposition

Following the presentation in [11], we first introduce notation for indices. In general, any subset of $\{1, \ldots, M\}$ denotes an index. For an index $t \subseteq \{1, \ldots, M\}$, $|t|$ denotes the cardinality of t. For the special case that $t = \emptyset$, we define $|t| = 0$. For an index $t \neq \emptyset$, we sort its elements in ascending order and express it as $t = (t_1, \ldots, t_{|t|})$ with $t_1 < t_2 \ldots < t_{|t|}$. In addition, we also call $|t|$ the (ANOVA) order of t, and call t a $|t|$-th order index. For a given ANOVA order $i = 0, \ldots, M$, we define the following index sets

$$\mathfrak{T}_i := \{t \,|\, t \subset \{1, \ldots, M\},\ |t| = i\},$$

$$\mathfrak{T}_i^\star := \cup_{j=0,1,\cdots,i} \mathfrak{T}_j,$$

$$\mathfrak{T} := \mathfrak{T}_M^\star = \cup_{j=0,1,\cdots,M} \mathfrak{T}_j.$$

The sizes of the above sets (numbers of elements that they contain) are denoted by $|\mathfrak{T}_i|$, $|\mathfrak{T}_i^\star|$ and $|\mathfrak{T}|$ respectively. From the above definition, $\mathfrak{T}_0 = \{\emptyset\}$ and $|\mathfrak{T}_0| = 1$ (since $\{\emptyset\}$ is not empty). For a given index $t = (t_1, \ldots, t_{|t|}) \in \mathfrak{T}$ with $|t| > 0$, ξ_t denotes a random vector collecting components of ξ associated with t, i.e., $\xi_t := [\xi_{t_1}, \ldots, \xi_{t_{|t|}}]^T \in I^{|t|}$, and we denote the probability density function of ξ_t by π_t.

While ANOVA methods for solving stochastic PDEs are discussed in detail in [12,14], in this paper we only focus on the anchored ANOVA method [40]. Given an anchor point $c = [c_1, \ldots, c_M]^T \in I^M$, the anchored ANOVA method decomposes the solution $u(x, \xi)$ of the global problem (1)–(2) as follows

$$\begin{aligned} u(x, \xi) &= u_0(x) + u_1(x, \xi_1) + \ldots + u_{1,2}(x, \xi_{1,2}) + \ldots \\ &= \sum_{t \in \mathfrak{T}} u_t(x, \xi_t), \end{aligned} \tag{3}$$

where we denote $u_\emptyset(x, \xi_\emptyset) := u_0(x)$ for convenience, and each term in (3) is specified as

$$u_\emptyset(x, \xi_\emptyset) := u_0(x) := u(x, c), \tag{4}$$

$$u_t(x, \xi_t) := u(x, c, \xi_t) - \sum_{s \subset t} u_s(x, \xi_s). \tag{5}$$

In (4), $u(x, c)$ is the solution of the deterministic version of (1)–(2) with the realization $\xi = c$, while $u(x, c, \xi_t)$ in (5) is the solution of a semi-deterministic version of (1)–(2) through fixing $\xi_i = c_i$ for $i \in \{1, \ldots, M\} \setminus t$, i.e.,