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High order finite volume WENO schemes for the Euler equations under gravitational fields

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ABSTRACT

Euler equations with gravitational source terms are used to model many astrophysical and atmospheric phenomena. This system admits hydrostatic balance where the flux produced by the pressure is exactly canceled by the gravitational source term, and two commonly seen equilibria are the isothermal and polytropic hydrostatic solutions. Exact preservation of these equilibria is desirable as many practical problems are small perturbations of such balance. High order finite difference weighted essentially non-oscillatory (WENO) schemes have been proposed in [22], but only for the isothermal equilibrium state. In this paper, we design high order well-balanced finite volume WENO schemes, which can preserve not only the isothermal equilibrium but also the polytropic hydrostatic balance state exactly, and maintain genuine high order accuracy for general solutions. The well-balanced property is obtained by novel source term reformulation and discretization, combined with well-balanced numerical fluxes. Extensive one- and two-dimensional simulations are performed to verify well-balanced property, high order accuracy, as well as good resolution for smooth and discontinuous solutions.

1. Introduction

Euler equations with gravitational source terms are widely used to model many interesting physical phenomena in the astrophysical and atmospheric science. These equations governing the conservation of mass, momentum and energy, coupled with source terms due to the gravitational field, are given by

$$
\begin{aligned}
&\rho_t + \nabla\cdot(\rho\mathbf{u}) = 0,\\
&(\rho\mathbf{u})_t + \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u} + p\mathbf{I}_d) = -\rho\nabla\phi, \qquad (1.1)\\
&E_t + \nabla\cdot((E+p)\mathbf{u}) = -\rho\mathbf{u}\cdot\nabla\phi,
\end{aligned}
$$

where $\mathbf{x}\in\mathcal{R}^d$ $(d=1,2,3)$ is the spatial variable, ρ denotes the fluid density, $\mathbf{u}$ is the velocity, p represents the pressure, and $E=\frac{1}{2}\rho\|\mathbf{u}\|^2+\rho e$ (e is internal energy) is the non-gravitational energy which includes the kinetic and internal energy of the fluid. $\phi=\phi(\mathbf{x})$ is the time independent gravitational potential. The operators ∇, $\nabla\cdot$ and $\otimes$ are the gradient, divergence and tensor product in $\mathcal{R}^d$, respectively, and $\mathbf{I}_d$ stands for the identity matrix. To close the system, the pressure p is linked to

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the density and the interval energy through an equation of state denoted by $p = p(\rho, e)$. The ideal gas law for the equation of state takes the form of

$$p = (\gamma - 1)\rho e = (\gamma - 1)\left(E - \rho\|\mathbf{u}\|^2/2\right), \tag{1.2}$$

where γ is the ratio of specific heats.

In one spatial dimension, the Euler equations (1.1) take the form of

$$\begin{aligned} &\rho_t + (\rho u)_x = 0, \\ &(\rho u)_t + (\rho u^2 + p)_x = -\rho\phi_x, \\ &E_t + ((E + p)u)_x = -\rho u \phi_x, \end{aligned} \tag{1.3}$$

with u being the velocity. Such model belongs to the general class of hyperbolic conservation laws with source terms, often referred as hyperbolic balance laws, which takes the general form of

$$U_t + f(U)_x = S(U, \phi), \tag{1.4}$$

where U is the solution vector with the corresponding flux $f(U)$, and $S(U, \phi)$ is the source term. The balance law usually admits non-trivial steady state solutions, in which the source term is exactly balanced by the flux gradient.

The Euler equations (1.3) under the static gravitation potential admit the hydrostatic stationary solution, also called mechanical equilibrium, where the velocity is zero and the external forces such as gravity are balanced by the pressure gradient force:

$$\rho = \rho(x), \quad u = 0, \quad p_x = -\rho\phi_x. \tag{1.5}$$

Two important special steady state arising in the applications are the constant temperature (isothermal) [22] and polytropic hydrostatic equilibrium states [9].

If the hydrostatic state is isothermal, we have $T(x) \equiv T_0 = const$ with $T(x)$ being the temperature. For an ideal gas satisfying

$$p(x) = \rho(x) R T(x), \tag{1.6}$$

where R is the gas constant, integrating the steady state solution (1.5) yields

$$\rho = \frac{p_0}{RT(x)} \exp\left(-\int_{x_0}^{x} \frac{\phi_x(s)}{RT(s)} ds\right), \qquad u = 0, \qquad p = p_0 \exp\left(-\int_{x_0}^{x} \frac{\phi_x(s)}{RT(s)} ds\right), \tag{1.7}$$

where p_0 is the initial pressure at some reference position x_0. Under the isothermal assumption, the equilibrium correspondingly becomes

$$\rho = \rho_0 \exp\left(-\frac{\phi}{RT_0}\right), \qquad u = 0, \qquad p = p_0 \exp\left(-\frac{\phi}{RT_0}\right), \tag{1.8}$$

with $p_0 = \rho_0 R T_0$.

The other polytropic hydrostatic equilibrium is characterized by

$$p = K\rho^{\nu}, \tag{1.9}$$

which will lead to the form of

$$\rho = \left(\frac{\nu - 1}{K\nu}(C - \phi)\right)^{\frac{1}{\nu-1}}, \qquad u = 0, \qquad p = \frac{1}{K^{\frac{1}{\nu-1}}}\left(\frac{\nu - 1}{\nu}(C - \phi)\right)^{\frac{\nu}{\nu-1}}, \tag{1.10}$$

where C, K and ν are all constants. A special case of ν is the ratio of specific heats γ.

The simplest encountered gravity is the linear gravitational potential field with $\phi_x = g$, and the corresponding isothermal and polytropic hydrostatic balances take the form of

$$\rho = \rho_0 \exp\left(\frac{-g\rho_0 x}{p_0}\right), \qquad u = 0, \qquad p = p_0 \exp\left(\frac{-g\rho_0 x}{p_0}\right), \tag{1.11}$$

and

$$p = p_0^{\frac{1}{\nu-1}}\left(p_0 - \frac{\nu - 1}{\nu} g\rho_0 x\right)^{\frac{\nu}{\nu-1}}, \qquad u = 0, \qquad \rho = \rho_0 \left(\frac{p}{p_0}\right)^{\frac{1}{\nu}}. \tag{1.12}$$