



# Analysis and accurate numerical solutions of the integral equation derived from the linearized BGKW equation for the steady Couette flow

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## ABSTRACT

The integral equation for the flow velocity  $u(x; k)$  in the steady Couette flow derived from the linearized Bhatnagar–Gross–Krook–Wendler kinetic equation is studied in detail both theoretically and numerically in a wide range of the Knudsen number  $k$  between 0.003 and 100.0. First, it is shown that the integral equation is a Fredholm equation of the second kind in which the norm of the compact integral operator is less than 1 on  $L^p$  for any  $1 \leq p \leq \infty$  and thus there exists a unique solution to the integral equation via the Neumann series. Second, it is shown that the solution is logarithmically singular at the endpoints. More precisely, if  $x = 0$  is an endpoint, then the solution can be expanded as a double power series of the form  $\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{n,m} x^n (\ln x)^m$  about  $x = 0$  on a small interval  $x \in (0, a)$  for some  $a > 0$ . And third, a high-order adaptive numerical algorithm is designed to compute the solution numerically to high precision. The solutions for the flow velocity  $u(x; k)$ , the stress  $P_{xy}(k)$ , and the half-channel mass flow rate  $Q(k)$  are obtained in a wide range of the Knudsen number  $0.003 \leq k \leq 100.0$ ; and these solutions are accurate for at least twelve significant digits or better, thus they can be used as benchmark solutions.

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## 1. Introduction

In small scale devices, such as microfluidics, gaseous flows may become highly nonequilibrium, hence it is required to solve kinetic equations (cf. reviews [1,2], monographs [3–8], and relevant references therein) or even molecular dynamics (cf. [9]). Kinetic equations evolve in phase space  $(\mathbf{x}, \boldsymbol{\xi})$ , where  $\boldsymbol{\xi}$  is the particle velocity, thus numerical solutions of kinetic equations are far more challenging and demanding than numerical solutions of hydrodynamic equations in physical space  $\mathbf{x}$ . In addition, the collision term in the Boltzmann equation is an integral operator in a five-dimensional space, evaluation of which requires considerable computational effort. To mitigate the difficulty due to the complexity of the collision integral, one often considers the linearized Boltzmann equation or simpler model kinetic equations, such as the Bhatnagar–Gross–Krook–Wendler (BGKW) equation [10,11]. However, the simplification of the collision term does not alleviate the challenging demand of computing the distribution function in phase space.

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There are several canonical flows which have been studied for a long time, including Kramers' flow, Poiseuille flow, and Couette flow; and there has been a systematic and persistent effort to obtain accurate numerical solutions of the kinetic equations corresponding to these flows [12–28]. Accurate solutions of these canonical flows have been used as benchmarks to test numerical schemes for more general flows [22–25,28].

For the aforementioned canonical flows, the linearized BGKW equation with the Maxwell diffusive boundary conditions [29,4] leads to Fredholm integral equations of the second kind with weakly singular kernels, and it has been a continuous effort to obtain accurate numerical solutions of the integral equations for the canonical flows [22–25,28,30–32]. However, it appears that some advanced modern techniques developed in the last few decades for solving integral equations with singularities at the endpoints have yet to be fully utilized in the community of rarefied gas dynamics (RGD). Indeed, accurately solving these integral equations requires at least two apparatuses: (a) high-order quadratures to evaluate the involved singular integrals and functions and (b) proper discretization schemes to capture the singularities of the solution at the endpoints. Currently, the popular approaches for solving these integral equations use the Meijer  $G$ -function to compute the singular kernels [25,28], apply *smooth* Gaussian quadrature to evaluate the *singular* integrals, and represent the solution with some *global* expansion of *smooth* functions or piecewise polynomials on equispaced subintervals. First, the use of Meijer  $G$ -function for the evaluation of the singular kernel function (e.g., the Abramowitz function in many cases) is very likely to suffer from catastrophic cancellation for large values of the argument, as observed previously [33]. Second, the direct application of a *smooth* Gaussian quadrature to evaluate the singular integrals either results in a very low accuracy or demands an excessively large number of nodes required for the quadrature. Third and finally, the representation of the solution by using *globally* smooth polynomial expansion or piecewise polynomials on *uniform, equispaced* subintervals makes it difficult to capture the singularities of the solution. Thus, the predominant methodologies currently employed in the RGD community fail to obtain high-precision benchmark solutions of these integral equations with a reasonable computational effort [33].

The shortcomings of the aforementioned approaches are especially evident and apparent in the case of obtaining high-precision benchmark solutions of the linearized BGKW equation for canonical flows [22–25,28]. For example, Loyalka and Thompson [25] use the Nyström method with Gauss–Kronrod quadrature to solve the integral equation for Kramers' problem. They are only able to obtain the flow velocity with a precision of about 7 digits even though the largest quadrature order is 1312 and a precision of 60 digits is required for the arithmetic computation. More recently, Yap and Sader [28] use the same technique with Gauss–Legendre quadrature to study steady and oscillating Couette flows, and with a large linear system of size  $12,800^2 = 163,840,000$  and the arithmetics with 30 digits, they are only able to obtain the velocity between 5 and 8 significant digits for the Knudsen number  $k$  between 0.01 and 100.0, respectively. While requiring a significant computational effort in terms of both the multiple-precision arithmetics and large size of quadratures, these popular approaches [25,28] can only attain a precision far less than the IEEE 64 bit double-precision.

Perhaps the most accurate result for the integral equation for the steady Couette flow to date is obtained by Li et al. [32] with the Chebyshev collocation method, in which the singular kernel function, i.e., the Abramowitz function, is approximated by its Chebyshev expansion [34], as opposed to the use of the Meijer  $G$ -function in the previous works [25,28]. By choosing a proper and accurate approximation for the singular kernel, the work of Li et al. [32] greatly improves the accuracy of the solution while significantly reduces the computational effort. However, the work of Li et al. [32] does not employ any advanced modern techniques for numerical solution of the integral equations with singularities at the endpoints, which is the task of the present work.

As mentioned previously, the Couette flow is one of the canonical flows which play an important role for understanding the transport phenomena in gaseous flows in micro-scales (cf., e.g., [6,9] and references therein). In particular, the Knudsen layer formed at a solid boundary is of great interest. In this paper, we will study the integral equation which is derived from the linearized BGKW equation for the steady Couette flow [3–8]:

$$u(x) - \frac{1}{k\sqrt{\pi}} \int_{-1/2}^{1/2} J_{-1}\left(\frac{|x-y|}{k}\right) u(y) dy = f(x), \quad (1a)$$

$$f(x) := \frac{1}{2\sqrt{\pi}} \left[ J_0\left(\frac{1/2-x}{k}\right) - J_0\left(\frac{1/2+x}{k}\right) \right], \quad (1b)$$

where the parameter  $k$  is the Knudsen number and  $J_n$  is the  $n$ -th order Abramowitz function defined by

$$J_n(x) = \int_0^\infty t^n e^{-t^2-x/t} dt, \quad n \geq -1. \quad (2)$$

It is easy to show that the kernel function  $J_{-1}(|x|)$  has both absolute value and logarithmic singularities at  $x = 0$ , that is,

$$J_{-1}(|x|) = \phi_{-1}(x) + g_{-1}(x)|x| + h_{-1}(x) \ln |x|, \quad (3)$$

where  $\phi_{-1}$ ,  $g_{-1}$ , and  $h_{-1}$  are functions admitting a power series representation about  $x = 0$ . Then, by its definition (1b), the function  $f(x)$  on the right-hand side of (1a) is logarithmically singular at both endpoints  $x = \pm 1/2$ .

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