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A high-order positivity-preserving single-stage single-step method for the ideal magnetohydrodynamic equations

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ABSTRACT

We propose a high-order finite difference weighted ENO (WENO) method for the ideal magnetohydrodynamics (MHD) equations. The proposed method is single-stage (i.e., it has no internal stages to store), single-step (i.e., it has no time history that needs to be stored), maintains a discrete divergence-free condition on the magnetic field, and has the capacity to preserve the positivity of the density and pressure. To accomplish this, we use a Taylor discretization of the Picard integral formulation (PIF) of the finite difference WENO method proposed in Christlieb et al. (2015) [23], where the focus is on a high-order discretization of the fluxes (as opposed to the conserved variables). We use the version where fluxes are expanded to third-order accuracy in time, and for the fluid variables space is discretized using the classical fifth-order finite difference WENO discretization. We use constrained transport in order to obtain divergence-free magnetic fields, which means that we simultaneously evolve the magnetohydrodynamic (that has an evolution equation for the magnetic field) and magnetic potential equations alongside each other, and set the magnetic field to be the (discrete) curl of the magnetic potential after each time step. In this work, we compute these derivatives to fourth-order accuracy. In order to retain a single-stage, single-step method, we develop a novel Lax–Wendroff discretization for the evolution of the magnetic potential, where we start with technology used for Hamilton–Jacobi equations in order to construct a non-oscillatory magnetic field. The end result is an algorithm that is similar to our previous work Christlieb et al. (2014) [8], but this time the time stepping is replaced through a Taylor method with the addition of a positivity-preserving limiter. Finally, positivity preservation is realized by introducing a parameterized flux limiter that considers a linear combination of high and low-order numerical fluxes. The choice of the free parameter is then given in such a way that the fluxes are limited towards the low-order solver until positivity is attained. Given the lack of additional degrees of freedom in the system, this positivity limiter lacks energy conservation where the limiter turns on. However, this ingredient can be dropped for problems where the pressure does not become negative. We present two and three dimensional numerical results for several standard test problems including a smooth Alfvén wave (to verify formal order of accuracy), shock tube problems (to test the shock-capturing ability of the scheme), Orszag–Tang, and

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cloud shock interactions. These results assert the robustness and verify the high-order of accuracy of the proposed scheme.

1. Introduction

The ideal magnetohydrodynamic (MHD) equations model the dynamics of a quasi-neutral, perfectly conducting plasma [1]. A vast range of areas including astrophysics and laboratory plasmas can be modeled with this system. Among other methods, various high-order numerical schemes based on the essentially non-oscillatory (ENO) [2] as well as the weighted ENO (WENO) reconstruction technique [3–9] have been applied successfully to ideal MHD in the past two decades. These high order schemes are capable of resolving complex features such as shocks and turbulences using fewer grid points than low-order schemes for the same level of error, as is common with many high-order shock-capturing schemes.

It often happens in large-scale MHD simulations that the complex features are concentrated in a small portion of the simulation domain. Adaptive mesh refinement (AMR) is a technique that is designed for treating such locality of complexity in hydrodynamics and magnetohydrodynamics. One of the chief difficulties with implementing high order schemes within an AMR framework is that boundary conditions for the refined region need to be specified in a consistent manner [10]. This becomes difficult for multistage RK methods, because high order solutions cannot be found if one simply uses high-order interpolated values (in time) at the ghost points that are required for the intermediate stages of the method. Preliminary work that combines WENO spatial discretizations with strong stability preserving Runge–Kutta (SSP-RK) time-stepping is conducted in [11,12], and very recent work makes use of curvilinear grids to extend finite difference methods to problems with geometry [9]. However, the authors in [9] use global time steps (which precludes the possibility of introducing local time stepping), and perhaps more troublesome, they drop mass conservation for their framework to work.

In choosing building blocks for AMR code, it has been argued that single-stage, single-step methods are advantageous [13,14], partly because fewer synchronizations are needed per step than multistage RK methods. The fact that single-stage, single-step methods do not have an issue with these synchronizations is possibly one of the reasons they have gained much attention in the past two decades, and one reason we are choosing to pursue these methods. Broadly construed, these methods are based on Lax and Wendroff's original idea of using the Cauchy–Kovalevskaya procedure to convert temporal derivatives into spatial derivatives in order to define a numerical method [15]. Notable high-order single-stage, single-step methods include the Arbitrary DERivative (ADER) methods [16,17], the Lax–Wendroff finite difference WENO methods [18], the Lax–Wendroff discontinuous Galerkin (DG) methods [19] and space–time schemes applied directly to second-order wave equations [20,21]. Of the three classes of high-order methods based upon Lax–Wendroff time stepping, only the ADER methods have been applied to magnetohydrodynamics [14,7,22], whereas similar investigations have not been done for the other classes. An additional advantage that single-stage single-step Taylor methods offer is their *low-storage* opportunities. This requires care, because these methods can easily end up requiring the same amount of storage as their equivalent RK counterpart (e.g., if each time derivative is stored in order to reduce coding complexity).

The current work is based on the Taylor discretization of the Picard integral formulation of the finite difference WENO (PIF-WENO) method [23]. Compared with other WENO methods that use Lax–Wendroff time discretizations [18], our method has the advantage that its focus is on constructing high-order Taylor expansions of the fluxes (which are used to define a conservative method through WENO reconstruction) as opposed to the conserved variables. This allows, for example, the adaptation of a positivity-preserving limiter, which we describe in this document.

An important issue in simulations of MHD systems is the controlling of the divergence error of the magnetic field, since numerical schemes based on the transport equations alone will, in general, accumulate errors in the divergence of the magnetic field. Failure to address this issue creates an unphysical force parallel to the magnetic field [24], and if this is not taken care of, it will often lead to failure of the simulation code. Popular techniques used to solve this problem include (1) the non-conservative eight-wave method [25], (2) the projection method [24], (3) the hyperbolic divergence cleaning method [26], and (4) the various constrained transport methods [27,28,8,29–34]. Tóth conducts an extensive survey in [35].

The current paper uses the unstaggered constrained transport framework proposed by Rossmanith [34]. This framework evolves a vector potential that sits on the same mesh as the conserved quantities. This vector potential is then used to correct the magnetic field. Historically, the term "constrained transport" has been used to refer to a class of methods that incorporates the divergence-free condition into the discretization of the transport equation of the magnetic field, often done in a way that can be interpreted as maintaining an electric field on a staggered mesh [30,35]. Some authors actually still distinguish between this type of "constrained transport" and the "vector potential" approach [36]. However, as is well-known, evolving a vector potential is conceptually equivalent to evolving an electric field. The unstaggered approach has the added benefit of ease for potential embedding in an AMR framework.

An important piece in any vector-potential based constrained transport method is the discretization of the evolution equation of the vector potential. This evolution equation is a nonconservative weakly hyperbolic system, and it can be treated numerically from this viewpoint [33]. An alternative approach is to view it as a modified system of Hamilton–Jacobi