Journal of Computational Physics 315 (2016) 323–347

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

Bound-preserving discontinuous Galerkin methods for relativistic hydrodynamics

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ARTICLE INFO

Article history:
Received 14 April 2015
Received in revised form 16 February 2016
Accepted 16 February 2016
Available online 30 March 2016

Keywords:
Bound-preserving
Discontinuous Galerkin method
Relativistic hydrodynamics
L^1-stability

ABSTRACT

In this paper, we develop a discontinuous Galerkin (DG) method to solve the ideal special relativistic hydrodynamics (RHD) and design a bound-preserving (BP) limiter for this scheme by extending the idea in X. Zhang and C.-W. Shu, (2010) [56]. For RHD, the density and pressure are positive and the velocity is bounded by the speed of light. One difficulty in numerically solving the RHD in its conservative form is that the failure of preserving these physical bounds will result in ill-posedness of the problem and blowup of the code, especially in extreme relativistic cases. The standard way in dealing with this difficulty is to add extra numerical dissipation, while in doing so there is no guarantee of maintaining the high order of accuracy. Our BP limiter has the following features. It can theoretically guarantee to preserve the physical bounds for the numerical solution and maintain its designed high order accuracy. The limiter is local to the cell and hence is very easy to implement. Moreover, it renders L^1-stability to the numerical scheme. Numerical experiments are performed to demonstrate the good performance of this bound-preserving DG scheme. Even though we only discuss the BP limiter for DG schemes, it can be applied to high order finite volume schemes, such as weighted essentially non-oscillatory (WENO) finite volume schemes as well.

1. Introduction

Relativistic flows are widely used to model high-energy astrophysical phenomena, such as blast waves of supernova explosions, gravitational collapse and accretion, superluminal jets and gamma-ray bursts. When the speed of the flow is near the speed of the light but there is no strong gravitational field involved, the framework of special relativity is accurate to certain extent to describe the physical phenomena. In this paper, we discuss discontinuous Galerkin (DG) methods to solve the two-dimensional special relativistic hydrodynamics, which can be written into a system of conservation laws as below

$$\mathbf{w}_t + \mathbf{f}(\mathbf{w})_x + \mathbf{g}(\mathbf{w})_y = 0, \tag{1.1}$$

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1 Research supported by AFOSR grant F49550-12-1-0399 and NSF grant DMS-1418750.

http://dx.doi.org/10.1016/j.jcp.2016.02.079
0021-9991/

with

$$\mathbf{w}=\begin{pmatrix} D \\ m \\ n \\ E \end{pmatrix}, \quad \mathbf{f}(\mathbf{w})=\begin{pmatrix} Du \\ mu+p \\ nu \\ m \end{pmatrix}, \quad \mathbf{g}(\mathbf{w})=\begin{pmatrix} Dv \\ mv \\ nv+p \\ n \end{pmatrix} \tag{1.2}$$

as well as its one dimensional version. The method can be easily extended to three-dimensions, but this is not discussed in the paper. In (1.2), p, D, m, n and E are the thermal pressure, mass density, momentum in the x-direction, momentum in the y-direction, and energy, respectively. (u, v) is the velocity field of the fluid. Moreover, units are normalized such that the speed of light is $c = 1$. If we denote ρ to be the proper rest-mass density, then the conservative variable $\mathbf{w}$ can be written as

$$D = \gamma\rho, \tag{1.3}$$

$$m = Dh\gamma u, \tag{1.4}$$

$$n = Dh\gamma v, \tag{1.5}$$

$$E = Dh\gamma - p, \tag{1.6}$$

where $\gamma = (1 - u^2 - v^2)^{-1/2}$ is the Lorentz factor and h is the specific enthalpy. To close the system, we specify an equation of state $h = h(p, \rho)$. For ideal gas

$$\rho h = \rho + p\Gamma/(\Gamma - 1) \tag{1.7}$$

with Γ being the specific heat ratio, such that $1 < \Gamma \le 2$ (see for example [39]). Moreover, the sound speed is defined as

$$c_s = \sqrt{\frac{\Gamma p}{\rho h}} = \sqrt{\frac{(\Gamma - 1)(h - 1)}{h}}.$$

Physically, the density D and pressure p are positive, and the velocity field (u, v) satisfies $u^2 + v^2 \le 1$. Therefore, we define the admissible set to be

$$G = \{\mathbf{w} : D > 0, p(\mathbf{w}) > 0, u(\mathbf{w})^2 + v(\mathbf{w})^2 \le 1\}.$$

By (1.7), it is easy to see that $h > 1$, which further yields $0 < c_s \le 1$. It is demonstrated in [24] that G is convex and can be represented as

$$G = \{\mathbf{w} : D > 0, E > \sqrt{D^2 + m^2 + n^2}\}. \tag{1.8}$$

In order to update the flux in the computation, we further need the inverse of (1.3)–(1.6) from the conservative vector $\mathbf{w}$ to the primitive vector $\mathbf{u} = \{\rho, u, v, p\}$. Unlike its Newtonian counterpart, in RHD we do not have an explicit formula for this inverse map. By (1.6), (1.7) and the definition of γ, we can derive the nonlinear equation satisfied by the pressure p

$$f(p; \mathbf{w}) := E - \frac{p}{\Gamma - 1} - D\sqrt{1 - \frac{m^2 + n^2}{(E + p)^2}} - \frac{m^2 + n^2}{E + p} = 0, \qquad p \in [0, +\infty) \tag{1.9}$$

By a simple calculation one can show that, if $\mathbf{w} \in G$, then $\frac{\partial f}{\partial p}(p; \mathbf{w}) < 0$ and the equation (1.9) has a unique positive solution [47]. In practice, once the bounds (1.8) are satisfied by the numerical solution, (1.9) can be solved efficiently by standard root finding methods. After the pressure is obtained, other quantities can be calculated sequentially and directly via (1.3)–(1.7). Therefore, an efficient and effective conversion from the conservative vector to the primitive vector crucially depends on guaranteeing the bound-preserving property (1.8) for the numerical solution, which is the main objective of this paper.

Numerical simulation of RHD has been intensively studied in the last few decades. The first Eulerian method dates back to the early work by Wilson [44,45], in which the author used explicit finite differencing techniques and a monotonic transport algorithm to discretize the advection terms of the RHD equations. They applied the Von Neumann–Richtmyer artificial viscosity method [26,33] to handle the shock waves. Though this procedure remained standard through the 1980s, it turned out to be unable to resolve the extremely strong shock structures that would appear in the ultra-relativistic regime ($\gamma \ge 2$) [1]. A major breakthrough came in 1994 when Martí et al. [19,18,20] reformulated the RHD into the conservation form and for the first time introduced the Godunov-type *high resolution shock capturing* (HRSC) techniques from the classical gas dynamics simulation to that of the RHD. The HRSC techniques produced high-order approximation in the smooth region and were able to capture shocks and steep transients sharply without spurious oscillations. Since then, various Riemann solvers and modern techniques in gas dynamics were extended to the RHD simulation, for example the relativistic Roe solver by Eulderink et al. [11,12], the HLLE solver extended by Schneider et al. [34] and the recent relativistic HLLC solver carried out by Mignone et al. [24]. Furthermore, Martí and Müller [21] extended the PPM method [6] to the one dimensional