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A priori grid quality estimation for high-order finite differencing

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ABSTRACT

Structured grids using the finite differencing method contain two sources of grid-induced truncation errors. The first is dependent on the solution field. The second is related only to the metrics of the grid transformation. The accuracy of the grid transformation metrics is affected by the inverse metrics, which are spatial derivatives of the grid in the generalised coordinates. The truncation errors contained in the inverse metrics are generated by the spatial schemes. Fourier analysis shows that the dispersion errors, by spatial schemes, have similarities to the transfer function of spatial filters. This similarity is exploited to define a grid quality metric that can be used to identify areas in the mesh that are likely to generate significant grid-induced errors. An inviscid vortex convection benchmark case is used to quantify the correlation between the grid quality metric and the solution accuracy, for three common geometric features found in grids: abrupt changes in the grid metrics, skewness, and grid stretching. A strong correlation is obtained, provided that the grid transformation errors are the most significant sources of error.

1. Introduction

There is a requirement in computational fluid dynamics (CFD) and computational aeroacoustic (CAA) simulations for low dispersion and low dissipation numerical methods to accurately predict the generation, propagation, and interaction of acoustic, entropic and vortical disturbances. This requirement can be satisfied by using high-order finite differencing (FD) methods, which have been applied in various CFD and CAA studies [1,2]. Finite-differencing schemes are derived from a Taylor series and contain truncation errors. The amplitude of the truncation error may depend on the solution field and on the grid coordinates. The latter is the source of error that relates to the grid quality. Grids of higher quality generate smaller truncation errors, and contribute less to the total error, thereby providing a more accurate solution.

Abrupt changes in the grid spacing or grid line direction, for example along block interfaces, have been identified as sources of grid-induced errors [3–5] and can lead to inaccuracies in the solution. Simple grid quality measures based only on local geometric properties have been suggested [6,7]. However, in these examples a monotonic and strong correlation between the mesh quality metric and the solution accuracy was not obtained. The lack of a strong correlation may be caused by the presence of additional measures to ensure numerical stability. For example, Visbal and Giatonde [5] showed that low-pass filters, which can have similar effects on a numerical solution to artificial dissipation, can provide stable and

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accurate solutions in areas of the mesh where there are abrupt changes in the grid spacing or grid line direction. Solutions obtained under a strong influence of additional filtering or dissipation may reduce the impact of poor grid quality. However, stronger filters remove fine scale features from the solution field, thereby reducing the fidelity of the solution. Some filtering is inherently required for high-order methods to remove spurious modes. However, Colonius and Lele [8] emphasise that the removal of spurious waves should be attempted by improving the grid quality and enhancing the boundary condition accuracy rather than by using stronger filters or additional dissipation.

Many body-fitted curvilinear grids are generated in a two-step procedure. Firstly, the grid points along the block edges are mapped by a distribution function. The parameters of the distribution function may be manually specified, which may impact geometric properties such as the local grid spacing, and the grid stretching. In the second step the mesh enclosed by the block edges is generated. This may be obtained by transfinite interpolation (TFI), or by elliptic solvers. The latter method ensures a smooth grid, and therefore smooth grid transformation metrics [9]. Elliptic solvers may be coupled with additional control functions (which have manually tunable parameters) to achieve desirable grid clustering properties [9,10]. The high level of manually specified parameters in the grid generation procedure, makes the grid quality prone to human errors. Therefore, an effective grid quality metric is required, in order to identify and resolve areas of poor grid quality efficiently. This is especially useful for manually generated structured, body fitted meshes typically used in high order finite difference CAA codes.

The quality of the mesh is often attributed to specific geometric properties such as the grid non-orthogonality [11,12], or the aspect ratio [7,13]. For any structured grid, all geometric features can be expressed by the grid point distribution function. Present grid quality measures, that depend on this function, are based on the work by Vinokur [14], Mastin [3] and Thompson et al. [15]. In these methods, the effects of the grid geometry on the overall truncation error are evaluated by numerically approximating the leading order terms of the truncation error series. For example, Lee and Tsuei [11] applied this method to derive an equation to estimate the truncation errors of the convection terms in the two-dimensional Navier–Stokes equations. The equation requires the numerical evaluation of second and third derivatives in space by additional differencing schemes and is non-trivial to solve. Additionally, the equation can only be evaluated once a flow field solution is obtained. The formula does however account for the additional errors generated when applying different differencing schemes for the spatial derivatives and the grid metrics. Deng et al. [16] showed, for governing equations expressed in a strong conservation form, that this inconsistency of spatial schemes can violate the surface conservation law. This violation may generate artificial and undesirable source and sink terms in the governing equations that result in numerical instabilities, and degrade the robustness of high-order methods.

In the current work an alternate approach to truncation error analysis is outlined. The truncation errors for a generic spatial scheme are expressed by Fourier analysis, and are shown to hold similarities to the transfer function of spatial filters. This similarity is exploited to define a grid quality metric scalar that correlates to the truncation errors contained in the grid metrics. The proposed grid quality metric is applied on several grids containing three commonly found geometric features: abrupt changes in the grid metrics, grid skewness, and grid stretching. The results from these tests are used to determine the correlation between the grid quality metric and the solution error.

2. Derivation of the grid quality metric

The methodology to derive the grid quality metric is described for a 1-D case. The application to higher dimensions is analogous. The 1-D wave equation in the generalised coordinate system is given by,

$$\frac{\partial f}{\partial t} + U\frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} = 0, \tag{1}$$

where f is a scalar variable, t is time, U is the convection speed in the x-direction, and ξ is the generalised coordinate. Equation (1) can be evaluated by spatial differencing schemes, and a temporal scheme, together with accurate boundary conditions. The term $\partial\xi/\partial x$ is the grid transformation metric that is evaluated by the inverse metric, which in turn is evaluated along the generalised coordinate. The first spatial derivative (D) of scalar field (f), with respect to ξ, can be evaluated by a generic spatial scheme expressed by,

$$D_i + \sum_{Z=-N}^{M} (\alpha_Z D_{i+Z}) = \frac{1}{\Delta\xi}\sum_{z=-n}^{m} (a_z f_{i+z}), \tag{2}$$

where the index i is the point on the grid line, $-N$ and M define the size of the implicit stencil, $-n$ and m define the range of the explicit stencil, and the coefficients α and a ensure the desired order of accuracy. Fourier analysis [17,18] can be applied to the spatial scheme to obtain,

$$jk^*\Delta\xi = \frac{\sum_{z=-n}^{m}\left(a_z e^{jk\Delta\xi z}\right)}{1+\sum_{Z=-N}^{M}\left(\alpha_Z e^{jk\Delta\xi Z}\right)}, \tag{3}$$

where $j=\sqrt{-1}$, and k and k^* are the original and modified wave numbers, respectively. This expression decomposes the truncation error of the spatial scheme into the contributions from discrete wave numbers. This approach to truncation error