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A biorthogonal decomposition for the identification and simulation of non-stationary and non-Gaussian random fields

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ABSTRACT

In this paper, a new method for the identification and simulation of non-Gaussian and non-stationary stochastic fields given a database is proposed. It is based on two successive biorthogonal decompositions aiming at representing spatio–temporal stochastic fields. The proposed double expansion allows to build the model even in the case of large-size problems by separating the time, space and random parts of the field. A Gaussian kernel estimator is used to simulate the high dimensional set of random variables appearing in the decomposition. The capability of the method to reproduce the non-stationary and non-Gaussian features of random phenomena is illustrated by applications to earthquakes (seismic ground motion) and sea states (wave heights).

1. Introduction

Natural hazards have to be accounted for when assessing the reliability of engineering structures such as dams, power plants, off-shore platforms, tunnels or bridges. Most of these phenomena, such as wind, earthquakes or sea waves, cannot be fully addressed through deterministic methods because of their inherent variability. As a consequence, it is a challenge to construct accurate and efficient stochastic models from data able to correctly reproduce their random features. However, the modelling of a non-stationary and non-Gaussian process is a difficult task, since it is represented through the time-dependent uncountable family of its marginal distributions. The task is even harder for stochastic fields [16,36], where the variables are multi-dimensional (*e.g.* time and space). This is why several authors have tackled part of this problem by developing methods dedicated to non-Gaussian but stationary [10,29,26,12] or to non-stationary but Gaussian processes [24]. In particular, a great number of methods proposed in the literature rely on the construction of a power spectral density to represent a Gaussian process or field, the simulation being realized using the spectral representation theorem. In this case, non-stationarity can be introduced through an evolutionary power spectral density [28,39]. This is why this approach is very popular and used in many engineering applications.

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In order to treat more general non-Gaussian processes in the framework of spectral representation, some authors propose to use a translation process [12,32,6]. This proved to be efficient also in strongly non-Gaussian cases but this approach is limited to the stationary case (although it has been extended in recent papers to the non-stationary case [31,17]). Moreover, the distribution of the random process has to be known which is not true in general. In consequence, these methods are not always adapted for addressing the simulation of a field given a database of realizations. Random phenomena generally feature a non-Gaussian, non-homogeneous and non-stationary behaviour and not only one or the other. This has been shown in many areas, such as oceanography [19,20], seismology [13,41] or wind engineering [21,24].

The Karhunen–Loève (KL) expansion is probably the most general approach to represent a non-Gaussian and non-stationary process [15,11,35,18,14,9,23,22]. In [41,25], the KL expansion is used to identify and simulate non-Gaussian and non-stationary stochastic processes from experimental data. In contrast to the Gaussian case, the random variables appearing in the decomposition are not Gaussian nor independent in the general case, but still uncorrelated. Their simulation, as detailed in [25], is realized through a kernel density estimator that takes into account the joint probability density of the whole random vector. In this paper we extend these works to the case of stochastic spatio–temporal fields. The proposed approach is closely related to the general framework of biorthogonal decomposition (BOD) introduced by Aubry et al. [1,2] for spatio–temporal signals. This approach consists in a proper orthogonal decomposition (POD) for spatio–temporal analysis, which is popular for the study of turbulence [34] and has been extended to stochastic fields by Venturi [37]. However, the direct computation of the multi-dimensional Karhunen–Loève expansion is generally not feasible due to its numerical cost induced by the numerical resolution of a Fredholm integral equation that corresponds to a high dimensional eigenvalue problem [27,5].

Based on these considerations, we propose a new general decomposition that allows to construct a multi-dimensional KL expansion even for reasonably large problems. It requires very few assumptions on the modelled field and has the advantage that the non-stationary and non-Gaussian features of the data are well reproduced, along with other important characteristics such as its spatial coherency. As it will be shown, its implementation is straightforward, and its simulation is fast so that it is suitable to use in a Monte Carlo method for example. The paper is organized as follows: first, we present the construction of the model and discuss some practical aspects of its implementation. Then, we present applications on a realistic earthquake database and on wave data from fluid mechanics simulations.

2. A double spectral decomposition

In this Section, the theoretical framework of our approach is presented. The method is based on a double decomposition of a second order stochastic field $X(t,x)$ given a database of n independent realizations. We construct the operators that enable to exhibit appropriate basis functions and allow to perform a second decomposition in a natural way. We next discuss some numerical aspects of the construction of the model and its simulation. Moreover, it is shown in Appendix A that the proposed model, when built on a finite number of records n, converges almost surely to the random field $X(t,x)$ as $n \to +\infty$.

2.1. Theoretical framework

Let $\mathcal{D}$ be a bounded subset of $\mathbb{R}$ and $\mathcal{B}$ a bounded subset of $\mathbb{R}^d$, where $d \in \mathbb{N}^*$ is the space dimension. We consider a zero-mean second order field $X(t,x;\omega)$ defined on a complete probability space $(\Omega,\mathcal{A},P)$ and almost surely continuous, *i.e.* for almost all ω in Ω, $X(\cdot,\cdot;\omega) \in C^0(\mathcal{D}\times\mathcal{B},\mathbb{R})$.

The first idea that one may have is to discretize in space and time and to perform a Karhunen–Loève expansion, that is to compute the full auto-correlation $R(t,t';x,x') = \mathbb{E}\left(X(t,x)X(t',x')\right)$ and to solve the associated eigenvalue problem to obtain the classical Karhunen–Loève decomposition [15,41,25]:

$$X(t,x) = \sum_{k\geq 1} \sqrt{\lambda_k}\phi_k(t,x)\xi_k, \tag{1}$$

where $(\lambda_k)_{k\geq 1}$ are non-negative real numbers and $(\xi_k)_{k\geq 1}$ are real random variables. The main drawback of this approach is the size of the eigenvalue problem to be solved. Indeed, with N_t time steps and N_x points in each spatial direction, the discretized field may be viewed as the following discretized process $Z = \left(X(t_i,x_1)_{i=1}^{N_t},\ldots,X(t_i,x_{N_x})_{i=1}^{N_t}\right)$. The size of the eigenvalue problem to be solved is then $N_t \times N_x^d$, which is not treatable in most cases.

The key idea to overcome this problem is to use the isomorphism between $L^2(\mathcal{D}\times\mathcal{B}\times\Omega)$ and $L^2(\mathcal{D})\otimes L^2(\mathcal{B}\times\Omega)$ through the biorthogonal decomposition, as proposed in [1,37], and then to compute a second similar decomposition on $L^2(\mathcal{B}\times\Omega)$: we first isolate the temporal part of the field, and then separate the spatial and the random parts.

Let us begin with the first decomposition. Let $H_1 = L^2(\mathcal{D})$ and $H_2 = L^2(\mathcal{B}\times\Omega)$, with the following inner products:

$$\begin{cases} \forall\, \phi_1,\ \phi_2 \in H_1, & (\phi_1,\phi_2)_{H_1} = \int_{\mathcal{D}} \phi_1(t)\phi_2(t)\,\mathrm{d}t, \\ \forall\, v_1,\ v_2 \in H_2, & (v_1,v_2)_{H_2} = \int_{\mathcal{B}} \mathbb{E}\left(v_1(x)v_2(x)\right)\,\mathrm{d}x, \end{cases}$$

where we omitted the dependency of v_1, v_2 on the random event ω. In order to perform a first decomposition, we define $U : H_1 \to H_2$,