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Nash equilibrium and multi criterion aerodynamic optimization ☆

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ABSTRACT

Game theory and its particular Nash Equilibrium (NE) are gaining importance in solving Multi Criterion Optimization (MCO) in engineering problems over the past decade. The solution of a MCO problem can be viewed as a NE under the concept of competitive games. This paper surveyed/proposed four efficient algorithms for calculating a NE of a MCO problem. Existence and equivalence of the solution are analyzed and proved in the paper based on fixed point theorem. Specific virtual symmetric Nash game is also presented to set up an optimization strategy for single objective optimization problems. Two numerical examples are presented to verify proposed algorithms. One is mathematical functions' optimization to illustrate detailed numerical procedures of algorithms, the other is aerodynamic drag reduction of civil transport wing fuselage configuration by using virtual game. The successful application validates efficiency of algorithms in solving complex aerodynamic optimization problem.

1. Introduction

Multi criterion optimization can be described as a methodology for the design of systems where the interaction between several criteria must be considered and where the designer is free to significantly affect the system performance in more than one criterion. In the literature, contributions to single-discipline and/or single-point design optimization abound. The two main challenges of MCO are computational expense and organization complexity [1].

There are two commonly-used approaches to solve MCO problems. In the first one, which is well known in engineering, the different criteria are aggregated with different weights to form a single function to be minimized. However, when criteria arise from independent considerations, the aggregation lacks significance; additionally, the assignment of weights usually involves a lot of arbitrariness. The second approach is to identify the Pareto equilibrium front. The knowledge of the Pareto equilibrium front is a very rich and useful information for the designer. Generally, Evolutionary Algorithms (EAs) benefit from their robustness in capturing convex or non-convex, discrete or discontinuous Pareto fronts of MCO problems. Such EAs and Pareto front approaches have stimulated more and more investigations related to modern design problems raised by industry, despite the fact that engineers are often reluctant to use this very time-consuming evolutionary process.

However, another multiple objective scheme, this time a non-cooperative one, has been presented by J. Nash in the early 50s [2,3]. This approach introduced the notion of player and aimed at solving multiple objective optimization problems

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originating from game theory and economics. The contribution of John Forbes Nash in his 1951 article *Non-Cooperative Games* was to define a mixed strategy NE for any game with a finite set of actions and prove that at least one (mixed strategy) NE must exist in such a game. The NE is a solution concept of a game involving two or more players, in which each player is assumed to know the equilibrium strategies of the other players, and no player has anything to gain by changing only his own strategy unilaterally [4]. If each player has chosen a strategy and no player can benefit by changing his or her strategy while the other players keep theirs unchanged, then the current set of strategy choices and the corresponding payoffs constitute a NE.

Game theorists use the NE concept to analyze the outcome of the strategic interaction of several decision makers. In other words, it provides a way of predicting what will happen if several people or several institutions are making decisions at the same time, and if the outcome depends on the decisions of the others. The simple insight underlying John Nash's idea is that we cannot predict the result of the choices of multiple decision makers if we analyze those decisions in isolation. Instead, we must ask what each player would do, taking into account the decision-making of the others. The answer to this question is very closely related to the nature of the models and of the objective functionals that are considered.

NE has been used in variance areas, such as Prisoner's dilemma [5], Wardrop's principle [6], Tragedy of the Commons [7] and Matching pennies [8]. Over the past decade, NE theory became an efficient tool to solve MCO problems in aerodynamics [9,10] and other relative fields [11,12]. The solution of a MCO can be viewed as a NE under the concept of competitive games. This paper surveyed/proposed four popular algorithms for calculating the NE of a MCO problem, e.g. multi-criterion aerodynamic optimization. Existence and equivalence of the solution of algorithms are analyzed and proved based on fixed point theorem. In a competitive Nash game, each player attempts to optimize his own criterion through a symmetric exchange of information with others. A NE is reached when each player, constrained by the strategy of the others, cannot improve further its own criterion. Specific virtual symmetric Nash game is also implemented to set up an optimization strategy for solving single objective design problem. Two examples are presented to illustrate proposed algorithms. One is mathematical functions' optimization to indicate the detailed numerical procedure of algorithms, the other is aerodynamic drag reduction of civil transport wing fuselage configuration by using virtual game.

In the next section, we briefly present the basic knowledge from the game theory. Then, in Section 3, four efficient algorithms for calculating the NE are presented. In Section 4, existence and equivalence of the solution of algorithms are presented. Section 5 presents virtual symmetric Nash game and its corresponding algorithms. Section 6 solves an aerodynamic drag reduction of civil transport wing fuselage configuration by using virtual game. Finally a short conclusion is given in Section 7.

2. Nash equilibrium

Let N be the number of players. Each player $\nu \in \{1, \ldots, N\}$ controls the variables $x_\nu \in \mathbf{R}^{n_\nu}$. Let $x = (x_1, \ldots, x_N)^T \in \mathbf{R}^n$ be the vector formed by all these decision variables, where $n := n_1 + \ldots + n_N \geq N$. To emphasize the νth player's variables within the vector x, we sometimes write $x = (x_\nu, x_{-\nu})^T$, where $x_{-\nu}$ subsumes all the other players' variables.

Let $\mathcal{J}_\nu : \mathbf{R}^n \longmapsto \mathbf{R}$ be the νth player's payoff (or loss) function. We assume that these payoff functions are at least continuous, and we further assume that the functions $\mathcal{J}_\nu(x) = \mathcal{J}_\nu(x_\nu, x_{-\nu})$ are convex in the variable x_ν. In the classical Nash Equilibrium Problem (NEP), the variable x_ν belongs to a nonempty, closed and convex set $\mathbf{X}_\nu \subseteq \mathbf{R}^{n_\nu}$, $\nu = 1, \ldots, N$. Let

$$\mathbf{X} := \mathbf{X}_1 \otimes \ldots \otimes \mathbf{X}_N \tag{1}$$

be the Cartesian product of the strategy sets of each player.

Definition 1 *(Nash equilibrium problem).* Each player $\nu = 1, \ldots, N$, taking the other players' strategies $x_{-\nu}$ as exogenous variables, solves the minimization problem:

$$\begin{aligned} &P_\nu(x_{-\nu}) : \min_{x_\nu} \mathcal{J}_\nu(x_\nu, x_{-\nu}) \\ &\quad \textit{Subject to } x_\nu \in \mathbf{X}_\nu \subseteq \mathbf{R}^{n_\nu} \end{aligned} \tag{2}$$

where $\mathcal{J}_\nu$ is a given cost function of player ν.

Definition 2 *(Nash equilibrium).* A vector $x^\star \in \mathbf{X}$ is called a Nash equilibrium, or a solution of the NEP (2), if the block component $x^\star_\nu$ satisfies

$$\mathcal{J}_\nu(x^\star_\nu, x^\star_{-\nu}) \leq \mathcal{J}_\nu(x_\nu, x^\star_{-\nu}) \qquad \forall x_\nu \in \mathbf{X}_\nu \tag{3}$$

for all $\nu = 1, \ldots, N$.

Theorem 1 *(Nash, 1950). For any normal form game with a finite number of players (or player populations) each with a finite number of pure strategies, there exist a Nash equilibrium.*

Proposition 1 *(Property of A NE). A strategy pair $(x^\star_\nu, x^\star_{-\nu})$ is said to be a Nash equilibrium, then*