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Short note

A decoupled monolithic projection method for natural convection problems

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ABSTRACT

We propose an efficient monolithic numerical procedure based on a projection method for solving natural convection problems. In the present monolithic method, the buoyancy, linear diffusion, and nonlinear convection terms are implicitly advanced by applying the Crank–Nicolson scheme in time. To avoid an otherwise inevitable iterative procedure in solving the monolithic discretized system, we use a linearization of the nonlinear convection terms and approximate block lower–upper (LU) decompositions along with approximate factorization. Numerical simulations demonstrate that the proposed method is more stable and computationally efficient than other semi-implicit methods, preserving temporal second-order accuracy.

1. Introduction

Over the decades, the natural convection phenomenon has received considerable attention [1–9] because of a variety of real-world applications, such as mantle convection, processor cooling device, indoor ventilation, and solar convection zone. Since a typical Rayleigh number (*Ra*) is extremely large and ranges from 10^7 to 10^{24} depending on problem scales [1], it is important to investigate time-dependent dynamics in natural convection, relying on a strong coupling between incompressible flows and heat transfers. This coupling is related to the fact that the thermo-fluid flow is driven by a buoyancy force depending on temperature distribution while the temperature is convected by the background fluid flow. De Vhal Davis [2] numerically solved the natural convection problems based on stream function–vorticity formulations and provided numerical solutions for *Ra* ranging from 10^3 to 10^6 by using forward Euler discretization in time and second-order central difference in space. Quéré [4] obtained numerical solutions for *Ra* up to 10^8 with a pseudo-spectral Chebyshev algorithm and a temporally second-order advancement approach that combines the second-order backward difference formula for the linear diffusion terms with the Adams–Bashforth scheme for the buoyancy and nonlinear convection terms. Armfield et al. [6] investigated natural convection problems using a projection method that has been proven to be effective and widely used for incompressible fluid problems. This projection method is based on a semi-implicit discretization in time, in which the buoyancy and nonlinear convection terms are explicitly treated by applying the Adams–Bashforth scheme, and the linear diffusion terms are implicitly treated by applying the Crank–Nicolson scheme. The explicit treatments of the

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buoyancy and nonlinear convection terms allow a numerical decoupling between the momentum and energy equations [2,4,6], which leads to a severe restriction on the computational time step due to numerical instability.

Recently, in order to mitigate the time step restriction of the stability requirement, several researchers developed more stable and robust methods based on iterative monolithic procedures [7–9] for solving time-dependent natural convection problems. Zhang et al. [8] established two monolithic nonlinear projection-based numerical schemes with the backward Euler and Crank–Nicolson schemes for temporal discretization and stabilized mixed finite element spatial discretization, respectively. Deteix et al. [9] proposed a coupled prediction scheme based on a fixed-point iterative procedure in accordance with a projection method, whereas the Crank–Nicolson scheme is used for all temporal discretizations. As expected, these monolithic methods [8,9] admit a fairly large computational time step for obtaining stable numerical solutions. However, they require a time-consuming iterative procedure for solving coupled nonlinear discrete systems.

Inspired by the work of Kim et al. [10], who developed a non-iterative implicit projection scheme for solving incompressible Navier–Stokes equations, we extend this idea to natural convection problems. In this short note we propose a decoupled monolithic projection method (DMPM), preserving a temporal second-order accuracy. All terms in the momentum and energy equations are implicitly discretized, based on the Crank–Nicolson scheme along with linearization treatments for the nonlinear convection terms. We use approximate block LU decompositions of the coupled global operator matrix in the linear discretized system to obtain decoupled linear subsystems. All of the above treatments lead to a non-iterative monolithic procedure in the sense that momentum and energy equations are solved only once per time step, while other previously proposed monolithic schemes [7–9] for natural convection problems are mostly based on iterative approaches. The present DMPM, although it requires solving the Poisson equation twice per time step, allows a much larger computational time step than other semi-implicit methods, and thus saves significant computation time. Validation and performance of the proposed scheme are compared against the scheme that treats the nonlinear convection terms explicitly.

2. Construction of the DMPM

Under the Boussinesq approximation, governing equations for a three-dimensional (3D) natural convection flow can be written as

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + Pr\nabla^2\mathbf{u} + \mathbf{F}\theta, \tag{1}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{2}$$

$$\frac{\partial \theta}{\partial t} + \mathbf{u}\cdot\nabla\theta = \nabla^2\theta, \tag{3}$$

where $\mathbf{u} = (u_1, u_2, u_3)^T$, p, θ, and $\mathbf{F}\theta = (0, 0, PrRa\ \theta)^T$ are the velocity vector, pressure, temperature, and buoyancy force vector, respectively. In the equations above, the following non-dimensionalization is employed:

$$x_i = \frac{\widetilde{x}_i}{L},\quad u_i = \frac{\widetilde{u}_i}{\alpha/L},\quad t = \frac{\widetilde{t}}{L^2/\alpha},\quad p = \frac{\widetilde{p}}{\rho\alpha^2/L^2},\quad \theta = \frac{\widetilde{\theta}-\theta_c}{\theta_h-\theta_c},\quad Pr = \frac{\nu}{\alpha},\quad Ra = \frac{g\beta(\theta_h-\theta_c)L^3}{\nu\alpha},$$

where L, θ_h, θ_c, α, β, ν, ρ, and g represent the characteristic length, temperature of the hot wall, temperature of the cold wall, thermal diffusivity, thermal expansion coefficient, kinematic viscosity, fluid density, and gravitational acceleration, respectively, and the symbol $\widetilde{\ }$ indicates the dimensional variable.

Using a method similar to the basic rationale of the discretizations in Kim et al. [10], the governing equations in Eqs. (1)–(3) are discretized by using the Crank–Nicolson scheme for the buoyancy, nonlinear convection, and linear diffusion terms in time:

$$\begin{aligned}\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\triangle t} + \frac{1}{2}\left(\mathbf{u}^{n+1}\cdot\mathcal{G}\mathbf{u}^{n+1} + \mathbf{u}^n\cdot\mathcal{G}\mathbf{u}^n\right) &= -\mathcal{G}p^{n+1/2} + \frac{Pr}{2}\mathcal{L}\left(\mathbf{u}^{n+1}+\mathbf{u}^n\right)\\ &\quad + \frac{1}{2}\mathbf{F}\left(\theta^{n+1}+\theta^n\right) + \mathbf{mbc}^{n+1/2},\end{aligned} \tag{4}$$

$$\mathcal{D}\mathbf{u}^{n+1} = cbc^{n+1}, \tag{5}$$

$$\frac{\theta^{n+1}-\theta^n}{\triangle t} + \frac{1}{2}\left(\mathbf{u}^{n+1}\cdot\mathcal{G}\theta^{n+1} + \mathbf{u}^n\cdot\mathcal{G}\theta^n\right) = \frac{1}{2}\mathcal{L}\left(\theta^{n+1}+\theta^n\right) + ebc^{n+1/2}, \tag{6}$$

where $\mathcal{G}$, $\mathcal{L}$, and $\mathcal{D}$ represent discrete gradient, Laplacian, and divergence operators, respectively. The discrete spatial operators are evaluated using the second-order central difference scheme on a staggered grid. Here, $\triangle t$ is the time step, and the superscript n denotes the nth time level. Note that the boundary conditions of velocities have been incorporated into $\mathbf{mbc}^{n+1/2}$ and cbc^{n+1} [10], while $ebc^{n+1/2}$ contains the boundary conditions of both velocities and temperature. Moreover, preserving temporal second-order accuracy, the nonlinear convection terms in Eqs. (4) and (6) are linearized as

$$\frac{1}{2}\left(\mathbf{u}^{n+1}\cdot\mathcal{G}\mathbf{u}^{n+1} + \mathbf{u}^n\cdot\mathcal{G}\mathbf{u}^n\right) = \frac{1}{2}\left(\mathcal{N}\mathbf{u}^{n+1} + \mathbf{u}^{n+1}\cdot\mathbb{N}\right) + O\left(\triangle t^2\right), \tag{7}$$

$$\frac{1}{2}\left(\mathbf{u}^{n+1}\cdot\mathcal{G}\theta^{n+1} + \mathbf{u}^n\cdot\mathcal{G}\theta^n\right) = \frac{1}{2}\left(\mathcal{N}\theta^{n+1} + \mathbf{u}^{n+1}\cdot\mathcal{N}_\theta\right) + O\left(\triangle t^2\right), \tag{8}$$