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WLS-ENO: Weighted-least-squares based essentially non-oscillatory schemes for finite volume methods on unstructured meshes

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ABSTRACT

ENO (Essentially Non-Oscillatory) and WENO (Weighted Essentially Non-Oscillatory) schemes are widely used high-order schemes for solving partial differential equations (PDEs), especially hyperbolic conservation laws with piecewise smooth solutions. For structured meshes, these techniques can achieve high order accuracy for smooth functions while being non-oscillatory near discontinuities. For unstructured meshes, which are needed for complex geometries, similar schemes are required but they are much more challenging. We propose a new family of non-oscillatory schemes, called *WLS-ENO*, in the context of solving hyperbolic conservation laws using finite-volume methods over unstructured meshes. WLS-ENO is derived based on Taylor series expansion and solved using a weighted least squares formulation. Unlike other non-oscillatory schemes, the WLS-ENO does not require constructing sub-stencils, and hence it provides a more flexible framework and is less sensitive to mesh quality. We present rigorous analysis of the accuracy and stability of WLS-ENO, and present numerical results in 1-D, 2-D, and 3-D for a number of benchmark problems, and also report some comparisons against WENO.

1. Introduction

Many physical phenomena, such as waves, heat conduction, electrodynamics, elasticity, etc., can be modeled by partial differential equations. With the development of computer technology, many numerical methods have been designed to solve these kinds of problems over the past decades. Among these there are finite difference methods and their generalizations, finite volume methods, and finite element methods.

In this paper, we consider the problem of reconstructing a piecewise smooth function, in the context of finite volume methods for hyperbolic conservation laws. Given a geometric domain $\Omega \subseteq \mathbb{R}^d$, suppose u is a time-dependent piecewise smooth function over Ω, such as a density function. For any connected region τ, the d-dimensional conservation law can be written in the form

$$\int_{\tau} \frac{\partial u(\boldsymbol{x}, t)}{\partial t} d\boldsymbol{x} = -\int_{\partial\tau} \boldsymbol{F}(u) \cdot d\boldsymbol{a}, \tag{1}$$

where $\partial\tau$ is the boundary of τ, and $\boldsymbol{F}$ is a function of u, corresponding to the flux.

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A finite volume method solves the problem by decomposing the domain Ω into cells $\{\tau_i \mid i = 1, \dots, N\}$. Let $|\tau_i|$ denote the volume of τ_i and $\overline{u}_i(t) = \frac{1}{|\tau_i|}\int_{\tau_i} u(\boldsymbol{x}, t)\, d\boldsymbol{x}$, the average of u over τ_i. We obtain an equation

$$\frac{d\overline{u}_i(t)}{\partial t} = -|\tau_i| \int\limits_{\partial\tau_i} \boldsymbol{F}(u) \cdot d\boldsymbol{a}, \tag{2}$$

for each τ_i. The boundary integral requires using numerical quadrature for the flux. The integration of the flux requires reconstructing u from the cell averages $\overline{u}(t)$ in an accurate and stable fashion, and then evaluating the reconstruction at the quadrature points along the cell boundaries. For stability, $\boldsymbol{F} \cdot \boldsymbol{n}$ is typically replaced by a numerical flux, such as the Lax–Friedrichs flux,

$$\boldsymbol{F} \cdot \boldsymbol{n} = \frac{1}{2}\left[\left(\boldsymbol{F}\left(u^-\right) + \boldsymbol{F}\left(u^+\right)\right) \cdot \boldsymbol{n} - \alpha\left(u^+ - u^-\right)\right], \tag{3}$$

where u^- and u^+ are the values of u inside and outside the cell τ_i. The parameter α is a constant, and it should be an upper bound of the eigenvalues of the Jacobian of u in the normal direction.

In this context, we formulate the mathematical problem addressed in this paper as follows: Given the cell averages $\overline{u}_i$ of a piecewise smooth function $u(\boldsymbol{x})$ for cell $\tau_1, \tau_2, \dots, \tau_N$, let h_i be some length measure of cell τ_i. Find a polynomial approximation $\widetilde{u}_i(\boldsymbol{x})$ of degree at most $p-1$ over τ_i, such that

$$\|\widetilde{u}_i(\boldsymbol{x}) - u_i(\boldsymbol{x})\| = \mathcal{O}(h_i^p), \quad \boldsymbol{x} \in \tau_i. \tag{4}$$

In other words, $\widetilde{u}_i(\boldsymbol{x})$ is a pth order accurate approximation to $u(\boldsymbol{x})$ inside τ_i. In the context of hyperbolic conservation laws, $u(\boldsymbol{x})$ in (4) is equal to $u(\boldsymbol{x}, t)$ in (1) at a given t. For the facet between two cells, these reconstructions give us two values u^- and u^+, which can then be substituted into (3) to calculate the numerical flux. These reconstructions must be accurate, and also must lead to stable discretizations of the hyperbolic conservation laws when coupled with some appropriate time integration schemes, such as TVD Runge–Kutta schemes [1].

This reconstruction problem is decidedly challenging, because hyperbolic conservation laws can produce non-smooth solutions. An approximation scheme for smooth functions may lead to oscillations that do not diminish as the mesh is refined, analogous to the Gibbs phenomena. Such oscillations would undermine the convergence of the solutions. The ENO (Essentially Non-Oscillatory) and WENO (Weighted Essentially Non-Oscillatory) schemes [2–4] have been successful in solving this problem. In a nutshell, the WENO schemes use a convex combination of polynomials constructed over some neighboring cells, with higher weights for cells with smoother solutions and lower weights for cells near discontinuities. As a result, these methods can achieve high-order accuracy at smooth regions while being non-oscillatory near discontinuities. These reconstructions can be integrated into both finite volume and finite difference methods. With years of development, finite volume WENO schemes have been applied to both structured and unstructured meshes and higher dimensions [5–10]. Various attempts have been applied to improve the weights for WENO reconstruction [11–14]. Also, they have used WENO schemes in many applications, such as shock vortex interaction [15], incompressible flow problems [16], Hamilton–Jacobi equations [17], shallow water equations [18], etc.

Along the path of applying WENO schemes on unstructured meshes, tremendous effort has been made to improve the robustness of the schemes. Early attempts [5] work well for most unstructured meshes, but some point distributions may lead to negative weights and in turn make the schemes unstable. An extension was proposed in [7] to mitigate the issue, but it still had limited success over complicated geometries due to inevitably large condition numbers of their local linear systems. More recently, several different partition techniques were proposed to improve stability, such as [19], which uses a hybrid of two different reconstruction strategies to achieve better results. The technique was adopted in [20–22] for further development.

In this paper, we propose a new family of reconstruction methods over unstructured meshes. We refer to the schemes as *WLS-ENO*, or *Weighted-Least-Squares based Essentially Non-Oscillatory* schemes. Unlike the WENO scheme, our approach uses a generalized finite difference (GFD) formulation based on weighted least squares, rather than a weighted averaging of traditional finite differences. The GFD method is derived rigorously from Taylor series, and hence can deliver the same order of accuracy as traditional finite differences. In WLS-ENO, the convexity requirement is satisfied automatically, since the weights are specified *a priori*. These properties enable a more systematic way to construct non-oscillatory schemes. We will present the detailed derivation of the schemes and their robust numerical solution techniques. We also show that the schemes are often more accurate than WENO schemes near discontinuities and enable more stable PDE solvers when used in conjunction with total variation-diminishing time-integration schemes such as TVD Runge–Kutta. We report theoretical analysis in 1-D as well as experimental results in 1-D, 2-D, and 3-D.

The remainder of this paper is organized as follows. Section 2 reviews the ENO and WENO schemes, as well as some related background knowledge. Section 3 presents the derivation and numerical methods of the WLS-ENO schemes. Section 4 analyzes the accuracy and stability of the WLS-ENO schemes, and compares them against WENO and its previous generalization to unstructured meshes. Section 5 presents some numerical results and comparisons against some other methods. Finally, Section 6 concludes the paper with some discussions on future research directions.