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Time exponential splitting technique for the Klein–Gordon equation with Hagstrom–Warburton high-order absorbing boundary conditions

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ABSTRACT

Klein–Gordon equations on an unbounded domain are considered in one dimensional and two dimensional cases. Numerical computation is reduced to a finite domain by using the Hagstrom–Warburton (H-W) high-order absorbing boundary conditions (ABCs). Time integration is made by means of exponential splitting schemes that are efficient and easy to implement. In this way, it is possible to achieve a negligible error due to the time integration and to study the behavior of the absorption error. Numerical experiments displaying the accuracy of the numerical solution for the two dimensional case are provided. The influence of the dispersion coefficient on the error is also studied.

1. Introduction

Many applications such as weather prediction, underwater acoustic or earth geophysics concern with wave problems in unbounded domains.

We consider dispersive waves propagating in $(-\infty,\infty)\times[a,b]$, a two dimensional strip. The south and north boundaries of the strip are denoted by Γ_S and Γ_N. Inside the strip, we consider the Klein–Gordon equation,

$$\partial_t^2 u - c^2\nabla^2 u + s^2 u = f. \tag{1}$$

Here $c = c(x,y)$ is the given wave speed, $s = s(x,y)$ the medium dispersion coefficient and $f(x,y,t)$ is a given source.

Our results hold when, outside a compact region Ω_0, the speed c and the dispersion coefficient are constant and the source f vanishes. However, for the sake of simplicity, we only consider examples which satisfy this assumption in the whole strip $(-\infty,\infty)\times[a,b]$.

On the south and north boundaries we consider Neumann boundary conditions,

$$\partial_y u = 0, \quad \text{on } \Gamma_S \text{ and } \Gamma_N. \tag{2}$$

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Finally, we consider the initial conditions,

$$u(x, y, 0) = u_0(x, y), \quad \partial_t u(x, y, 0) = v_0(x, y), \tag{3}$$

which satisfy the boundary conditions on Γ_S and Γ_N, and vanish outside Ω_0.

The numerical approximations of these problems need to reduce the computation to a finite domain. Therefore, we truncate the infinite domain by introducing the west artificial boundary Γ_W, located at $x = x_W$, $a \le y \le b$, and the east artificial boundary Γ_E at $x = x_E$, $a \le y \le b$. We denote by Ω the computational domain bounded by $\Gamma_N \cup \Gamma_W \cup \Gamma_S \cup \Gamma_E$, such that $\Omega_0 \subset \Omega$.

The function u satisfies the Klein–Gordon equation (1) inside Ω, the Neumann boundary condition (2) on Γ_S and Γ_N, and the initial conditions (3) in Ω. It is necessary to define suitable artificial boundary conditions on the artificial boundaries Γ_W and Γ_E. For this, there are several possibilities, but we have focused on the so-called Absorbing Boundary Conditions, which are designed to produce small reflections inside the computational domain and to have local character. The ABCs are built in order to achieve, after the discretization, a stable, accurate, efficient and easy to implement scheme. There exists a wide literature on this subject, see the works [5,6,10,13–17] and the review papers [9,11,12,26].

As an alternative, it can be considered first a space discretization of the problem and then obtain ABCs for the discrete problem. We have worked in this sense for example in [2–4].

In this paper, we have considered the Hagstrom–Warburton high-order ABCs, a modified version of the Higdon ABCs [18,19]. These ABCs use auxiliary variables to avoid high derivatives in their formulation [16,17]. Arbitrary order of absorption P can be achieved by introducing P auxiliary variables ϕ_j, $j = 1, \ldots, P$, satisfying the recursive relations

$$\begin{aligned} &(a_0\partial_t + c\partial_x)u &&= a_0\partial_t\phi_1, \\ &(a_j\partial_t + c\partial_x)\phi_j &&= (a_j\partial_t - c\partial_x)\phi_{j+1}, \quad j = 1, \ldots, P, \\ &\phi_{P+1} = 0, \end{aligned} \tag{4}$$

in the vicinity of Γ_E. The parameters a_j have to be chosen. We consider $a_j = 1$ for all j, following the recommendation done in [8] and the remark in [13] in which the authors find this choice satisfactory in general. Other choices of the parameters could be considered, see for example [17] and Section 6 in this paper.

From the assumption that the initial conditions have compact support away from Γ_W and Γ_E, we have,

$$\phi_j(y, 0) = 0, \quad \partial_t\phi_j(y, 0) = 0 \quad \text{on} \quad \Gamma_W, \Gamma_E.$$

From (2) and the recursive relations (4), the conditions

$$\partial_y\phi_j(a, t) = \partial_y\phi_j(b, t) = 0,$$

can be deduced.

In [15], it is established that the H-W ABCs of order P given by (4) may be rewritten as

$$(\partial_t + c\partial_x)u = \partial_t\phi_1, \tag{5}$$

$$\partial_t^2\phi_1 = c^2\left(\frac{1}{2}\partial_y^2\phi_0 + \frac{1}{4}\partial_y^2\phi_1 + \frac{1}{4}\partial_y^2\phi_2\right) - s^2\left(\frac{1}{2}\phi_0 + \frac{1}{4}\phi_1 + \frac{1}{4}\phi_2\right), \tag{6}$$

$$\partial_t^2\phi_j = c^2\left(\frac{1}{4}\partial_y^2\phi_{j-1} + \frac{1}{2}\partial_y^2\phi_j + \frac{1}{4}\partial_y^2\phi_{j+1}\right) - s^2\left(\frac{1}{4}\phi_{j-1} + \frac{1}{2}\phi_j + \frac{1}{4}\phi_{j+1}\right), \; j = 2, \ldots, P, \tag{7}$$

$$u = \phi_0, \quad \phi_{P+1} = 0. \tag{8}$$

On the boundary Γ_W, the equation (5) has to be replaced by $(\partial_t - c\partial_x)u = \partial_t\phi_1$, but the other conditions (6)–(8) are the same.

The H-W high-order ABCs permit to approach the exact solution with an arbitrary order of absorption P. Increasing the order P causes only a linear rising of the computational cost. However, sometimes the accuracy of the numerical solution is limited by the space and time discretization error, above all for the two dimensional case.

The space discretization is reached by means of finite differences on a uniform grid, with fourth order inside the computational domain. For the time discretization, we propose a fourth order exponential splitting method which improves the computational efficiency of the time integration. The combination of the exponential splitting scheme with H-W ABCs is not a trivial matter and, as far as we know, it has not made before. The splitting choice is specific for the problem here considered and it involves to make several momentous decisions. We split the auxiliary variables and the function u on the spatial nodes and, we make a splitting in order to obtain an explicit and simple to implement method. We show the improvement by comparing our splitting scheme with the standard fourth-order four-stage Runge–Kutta method used in [17]. Useful overviews of splitting methods can be found in the review papers [7,22]. Other interesting references are [23,24].

We notice that it is possible to use higher order space discretizations, (cf. [15]), along with higher order exponential splitting methods designed similarly to the one used in this work.

The paper is organized as follows. Section 2 is devoted to introduce the spatial discretization and the time exponential splitting method which we propose, along with some properties of stability in the discrete energy norm which show the feasibility of this full discretization. For this, a simple one dimensional problem with periodic boundary conditions is used.