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# Summation-by-parts operators for correction procedure via reconstruction

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ABSTRACT

The correction procedure via reconstruction (CPR, formerly known as flux reconstruction) is a framework of high order methods for conservation laws, unifying some discontinuous Galerkin, spectral difference and spectral volume methods. Linearly stable schemes were presented by Vincent et al. (2011, 2015), but proofs of non-linear (entropy) stability in this framework have not been published yet (to the knowledge of the authors). We reformulate CPR methods using summation-by-parts (SBP) operators with simultaneous approximation terms (SATs), a framework popular for finite difference methods, extending the results obtained by Gassner (2013) for a special discontinuous Galerkin spectral element method. This reformulation leads to proofs of conservation and stability in discrete norms associated with the method, recovering the linearly stable CPR schemes of Vincent et al. (2011, 2015). Additionally, extending the skew-symmetric formulation of conservation laws by additional correction terms, entropy stability for Burgers' equation is proved for general SBP CPR methods not including boundary nodes.



## 1. Introduction

In the field of computational fluid dynamics (CFD), low-order methods are generally robust and reliable and therefore employed in practical calculations. The main advantage of high-order methods towards low-order ones is the possibility of considerably more accurate solutions with the same computing cost, but unfortunately they are less robust and more complicated. In recent years many researchers focus on this topic. There has been a surge of research activities to improve and refine high-order methods as well as to develop new ones with more favourable properties.

We consider in this paper the *correction procedure via reconstruction* (CPR) method using *summation-by-parts* (SBP) operators. The CPR combines the *flux reconstruction* (FR) approach developed by Huynh [13] and the *lifting collocation penalty* (LCP) by Wang and Gao [24].

Huynh [13] introduced the FR approach to high-order spectral methods for conservation laws in one space dimension and its extension to multiple dimensions using tensor products in 2007. For the case of one spatial dimension, the ansatz amounts to evaluating the derivative of a discontinuous piecewise polynomial function by using its straightforward derivative estimate together with a correction term. Wang and Gao [24] generalised the FR approach in 2009 to lifting collocation penalty methods on triangular grids. Later, the authors involved in the construction of these methods combined the names in the unifying framework of *correction procedure via reconstruction* (CPR) methods, see [14]. The CPR creates a framework

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unifying several high-order methods such as *discontinuous Galerkin* (DG), *spectral difference* (SD) and *spectral volume* (SV) methods. These connections were already pointed out and investigated in more detail in [1,4,28].

A linear (von Neumann) stability analysis of FR schemes was carried out already by Huynh [13] and in extended form by Vincent, Castonguay and Jameson [21]. A one-parameter family of linearly stable schemes in one dimension was discovered by the same authors [22] using an energy method and extended in 2015 to multiple-parameter families [23]. Extensions of the one-parameter family to advection–diffusion problems and triangular grids were published by the same groups [2,3,25].

The analysis of nonlinear stability for CPR methods is far more complex and not as advanced as in the linear case. First results are available in [16,26,27].

The application of *summation-by-parts* (SBP) operators in the CPR framework supplies a new perspective here.

In the context of *finite difference* (FD) methods, summation-by-parts operators with *simultaneous approximation terms* (SATs) provide a suitable way to derive stable schemes in a multi-block fashion enforcing boundary conditions in a weak way. They enable the imitation of manipulations of the continuous problem for the discrete method and are thus able to translate results like well-posedness to its discrete counterpart stability. Review articles from Svärd and Nordström [19] and Del Rey Fernández, Hicken, and Zingg [6] provide an insight in the development over the last decades and recent results. There is a strong connection of SBP operators with both skew-symmetric formulations of conservation laws as a means to prove conservation and stability [7], and quadrature rules [12]. Recently, Gassner et al. applied the SBP SAT framework to a particular *discontinuous Galerkin spectral element method* (DGSEM) to prove stability and discrete conservation for different systems of conservation laws, see inter alia [8,9,18,11]. Another extension of SBP operators has been presented by Del Rey Fernández et al. [5], based on a numerical setting and allowing general operators, connected with quadrature rules.

Here, we use the SBP framework in the general CPR setting. We are able to demonstrate all well-known properties, which have already been proven, but we can further extend the CPR method and show conservation and stability in a nonlinear case, see section 4.

The paper is organised as follows. The SBP and CPR frameworks will be briefly explained in section 2. In the next section, we apply SBP operators in CPR methods and revisit the results of Vincent et al. [22,23] for constant velocity linear advection.

In section 4, we focus on Burgers' equation and prove both discrete conservation and stability for a skew-symmetric formulation and Lobatto–Legendre nodes, revisiting the results of [8].

Additionally, we suggest a generalisation of the CPR method to get stability for a general SBP basis, extending the skew-symmetric formulation and being both provably stable and conservative. Numerical test cases are used to confirm the theoretical results. Finally, we discuss open problems and give an outlook on future work.

## 2. Existing formulations for SBP operators and CPR methods

Both *finite difference* (FD) SBP methods and CPR schemes are designed as semidiscretisations of hyperbolic conservation laws

$$\partial_t u + \partial_x f(u) = 0, \tag{1}$$

equipped with appropriate initial and boundary conditions.

### 2.1. SBP schemes

Traditionally, SBP operators are used in the FD framework. In one space dimension, a set of nodes including both boundary points of the element are used to represent the solution values. Extensions to multiple dimensions are performed via tensor products. To compute the semidiscretisation of (1), $f(u)$ is evaluated at each node and a difference operator is applied. The notation using vectors $\underline{u}$ for the solution values and the differentiation matrix $\underline{\underline{D}}$ is very common and results in a finite difference approximation $\underline{\underline{D}}\,\underline{f}$ of $\partial_x f$.

In order to be an SBP operator, the derivative matrix needs to be written as $D = P^{-1}Q$, $Q + Q^T = B = \operatorname{diag}(-1, 0, \dots, 0, 1)$, where $P$ is a symmetric and positive definite matrix with associated norm $\|\underline{u}\|_P^2 = \underline{u}^T \underline{\underline{P}}\,\underline{u}$, approximating the $L^2$ norm, see inter alia the review [19] and references cited therein. Boundary (both of the computational domain and between blocks) conditions are imposed weakly, using a *simultaneous-approximation-term* (SAT) formulation (see inter alia [7]), involving differences of desired and given values at boundary points. Thus, the SBP CPR methods described in the next chapter extend these schemes.

### 2.2. CPR methods

The FR approach in one space dimension described by Huynh [13] uses a nodal polynomial basis of order $p$ in the standard element $[-1, 1]$. All elements are mapped to this standard element and the computations are performed there. Extensions to multiple dimensions are performed via tensor products. The semidiscretisation of (1) (i.e. the computation of $\partial_x f(u)$) consists of the following steps, see also the review [14] and references cited therein:

- Interpolate the solution to the cell boundaries at $-1$ and $1$ (if these values are not already given as coefficients of the nodal basis).