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Variable-order fractional numerical differentiation for noisy signals by wavelet denoising

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ABSTRACT

In this paper, a numerical method is proposed to estimate the variable-order fractional derivatives of an unknown signal in noisy environment. Firstly, the wavelet denoising process is adopted to reduce the noise effect for the signal. Secondly, polynomials are constructed to fit the denoised signal in a set of overlapped subintervals of a considered interval. Thirdly, the variable-order fractional derivatives of these fitting polynomials are used as the estimations of the unknown ones, where the values obtained near the boundaries of each subinterval are ignored in the overlapped parts. Finally, numerical examples are presented to demonstrate the efficiency and robustness of the proposed method.

1. Introduction

In recent decades, fractional calculus has been successfully extended from pure mathematical theory to practical applications in several fields, such as visco-elastic materials [1,2], economics [3], statistical mechanics [4], as well as solid mechanics [5], etc. Meanwhile, the booming development of numerical methods for fractional calculus promotes the topic to go further. These methods include finite difference method [6], Laplace transform method [7], Adomian decomposition method [8], variational iteration method [9], fractional differential transform method [10], operational approach [11,12], as well as orthogonal functions methods including Block pulse functions [13], Bernstein polynomials [14], generalized fractional-order Legendre functions [15], Chebyshev wavelets [16], Legendre wavelets [17], etc.

Fractional order numerical differentiation is a relevant research branch in signal processing. Numerous effective numerical algorithms have been proposed [18–27]. With regard to a noisy signal, a filter is generally used to smooth the signal, and then the fractional derivative of the filtered signal is considered as a differentiator. In [20], the Digital Fractional Order Savitzky–Golay Differentiator (DFOSGD) was introduced and compared to some existing fractional order differentiators. Afterwards, the fractional order Jacobi differentiator and a method combining B-Spline functions with Tikhonov regularization were proposed in [21,22] and [23], respectively.

Currently, the topic of variable-order fractional calculus is becoming more and more attractive. A variety of works have been presented in [28–37]. In [28–30], the authors have built several mathematical models involving variable-order fractional calculus. Due to the existence of variable-order fractional differential and integral operators, the solutions of such kind of problems are always quite difficult to find. Consequently, efficient numerical techniques are necessary to develop.

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In [29,30], numerical methods based on finite difference scheme were presented to solve variable-order fractional differential equations. In [33,34], a distinctive class of numerical methods based on operational matrices of Bernstein polynomials were provided for variable-order fractional linear cable equations and fractional diffusion equations, respectively. Legendre wavelets functions were also adopted to numerically solve similar problems in [35]. Fractional spectral collocation methods were applied to solve linear and nonlinear variable-order fractional partial differential equations in [36]. Moreover, second-order approximations were proposed to estimate variable-order fractional derivatives with both algorithms and applications in [37]. On account of these works, it is convinced that the relative researches can be fairly promising.

Based on such research background, variable-order fractional numerical differentiation for noisy signals is considered in this paper. For this purpose, a noisy signal is processed by means of wavelet denoising. Then, fitting polynomials are adopted to approximate the denoised signal via the least square technique. Thus, the variable-order fractional numerical differentiation can be easily performed.

The outline of this paper is as follows: In Section 2, some necessary definitions and properties on variable-order fractional calculus and wavelet analysis are presented. In Section 3, the main processes of the proposed wavelet method are presented. In Section 4, some numerical examples are illustrated. Finally, some conclusions are given in Section 5.

2. Preliminaries

This section presents some definitions and properties on variable-order fractional calculus and wavelet analysis, which will be useful in this work.

2.1. Variable-order fractional derivatives

In this paper, variable-order Caputo fractional derivatives are considered, which are defined as follows.

Definition 2.1. (See [33].) Let $a \in \mathbf{R}$, and $f \in C^1(\mathbf{R})$, where $C^1(\mathbf{R})$ refers to the set of functions that admit a derivative which is continuous on $\mathbf{R}$. Then, the *variable-order Caputo fractional derivative* of f is defined as follows:

$$\forall t \geq a, \quad D_{a,t}^{\alpha(t)} f(t) = \frac{1}{\Gamma(1-\alpha(t))} \int_a^t (t-\tau)^{-\alpha(t)} f'(\tau) d\tau, \tag{1}$$

where $0 < \alpha(t) \leq 1$, and $\Gamma(\bullet)$ is the Gamma function.

Then, the following two properties can be deduced.

Property 2.1. *(See [33].) Let $f(t) = (t-a)^n$, with $0 < \alpha(t) \leq 1$ and $t \geq a$, then we have:*

$$D_{a,t}^{\alpha(t)} (t-a)^n = \begin{cases} \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha(t))} (t-a)^{n-\alpha(t)}, & n \neq 0, \\ 0, & n = 0. \end{cases} \tag{2}$$

Property 2.2. *Let $f_1, f_2 \in C^1(\mathbf{R})$, with $0 < \alpha(t) \leq 1$ and $t \geq a$, then we have:*

$$D_{a,t}^{\alpha(t)} \big(\lambda_1 f_1(t) + \lambda_2 f_2(t)\big) = \lambda_1 D_{a,t}^{\alpha(t)} f_1(t) + \lambda_2 D_{a,t}^{\alpha(t)} f_2(t). \tag{3}$$

2.2. Foundation theory of wavelet analysis

In [38], a quite systematic introduction on wavelet tour in signal processing is provided by S. Mallat. Based on this excellent monograph, the wavelet denoising method will be applied in this paper. For this purpose, we begin with giving the following two definitions.

Definition 2.2 *(Frame and Riesz bases).* (See [38].) A sequence $\{\phi_n\}_{n\in\Lambda}$ is called a *frame* of a Hilbert space $\mathbf{H}$, if there exist two constants $B \geq A > 0$ such that:

$$\forall f \in \mathbf{H}, \quad A\|f\|^2 \leq \sum_{n\in\Lambda} \big|\langle f, \phi_n\rangle\big|^2 \leq B\|f\|^2, \tag{4}$$

where Λ refers to a finite or infinite index set. If the frame $\{\phi_n\}_{n\in\Lambda}$ is linear independent, it is called *Riesz basis*.

Based on Definition 2.2, the fundamental concept of “Multiresolution approximation” in wavelet analysis can be introduced as follows.