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Embedded discontinuous Galerkin transport schemes with localised limiters

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ABSTRACT

Motivated by finite element spaces used for representation of temperature in the compatible finite element approach for numerical weather prediction, we introduce locally bounded transport schemes for (partially-)continuous finite element spaces. The underlying high-order transport scheme is constructed by injecting the partially-continuous field into an embedding discontinuous finite element space, applying a stable upwind discontinuous Galerkin (DG) scheme, and projecting back into the partially-continuous space; we call this an embedded DG transport scheme. We prove that this scheme is stable in L^2 provided that the underlying upwind DG scheme is. We then provide a framework for applying limiters for embedded DG transport schemes. Standard DG limiters are applied during the underlying DG scheme. We introduce a new localised form of element-based flux-correction which we apply to limiting the projection back into the partially-continuous space, so that the whole transport scheme is bounded. We provide details in the specific case of tensor-product finite element spaces on wedge elements that are discontinuous P1/Q1 in the horizontal and continuous P2 in the vertical. The framework is illustrated with numerical tests.

1. Introduction

Recently there has been a lot of activity in the development of finite element methods for numerical weather prediction (NWP), using continuous (mainly spectral) finite elements as well as discontinuous finite elements [11,33,10,15,12,27,4,1]; see [26] for a comprehensive review. A key aspect of NWP models is the need for transport schemes that preserve discrete analogues of properties of the transport equation such as monotonicity (shape preservation) and positivity; these properties are particularly important when treating tracers such as moisture. Discontinuous Galerkin methods can be interpreted as a generalisation of finite volume methods and hence the roadmap for the development of shape preserving and positivity preserving methods is relatively clear (see [6] for an introduction to this topic). However, this is not the case for continuous Galerkin methods, and so different approaches must be used. In the NWP community, limiters for CG methods have been considered by [25], who used first-order subcells to reduce the method to first-order upwind in oscillatory regions, and [13], who exploited the monotonicity of the element-averaged scheme in the spectral element method to build a quasi-monotone limiter.

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In this paper, we address the problem of finding suitable limiters for the partially continuous finite element spaces for tracers that arise in the framework of compatible finite element methods for numerical weather prediction models [7,8,31,29]. Compatible finite element methods have been proposed as an evolution of the C-grid staggered finite difference methods that are very popular in NWP. Within the UK dynamical core "Gung–Ho" project, this evolution is being driven by the need to move away from the latitude–longitude grids which are currently used in NWP models, since they prohibit parallel scalability [32]. Compatible finite element methods rely on choosing compatible finite element spaces for the various prognostic fields (velocity, density, temperature, etc.), in order to avoid spurious numerical wave propagation that pollutes the numerical solution on long time scales. In particular, in three dimensional models, this calls for the velocity space to be a div-conforming space such as Raviart–Thomas, and the density space is the corresponding discontinuous space. Many current operational forecasting models, such as the Met Office Unified Model [9], use a Charney–Phillips grid staggering in the vertical, to avoid a spurious mode in the vertical. When translated into the framework of compatible finite element spaces, this requires the temperature space to be a tensor product of discontinuous functions in the horizontal and continuous functions in the vertical (more details are given below). Physics/dynamics coupling then requires that other tracers (moisture, chemical species *etc.*) also use the same finite element space as temperature.

A critical requirement for numerical weather prediction models is that the transport schemes for advected tracers do not lead to the creation of new local maxima and minima, since their coupling back into the dynamics is very sensitive. In the compatible finite element framework, this calls for the development of limiters for partially-continuous finite element spaces. Since there is a well-developed framework for limiters for discontinuous Galerkin methods [3,5,6,14,16,34,17,35], in this paper we pursue the three stage approach of (i) injecting the solution into an embedding discontinuous finite element space at the beginning of the timestep, then (ii) applying a standard discontinuous Galerkin timestepping scheme, before finally (iii) projecting the solution back into the partially continuous space. If the discontinuous Galerkin scheme is combined with a slope limiter, the only step where overshoots and undershoots can occur is in the final projection. In this paper we describe a localised limiter for the projection stage, which is a modification of element-based limiters [24,21] previously applied to remapping in [23,20]. This leads to a locally bounded advection scheme when combined with the other steps.

The main results of this paper are:

1. The introduction of an embedded discontinuous Galerkin scheme which is demonstrated to be linearly stable.
2. The introduction of localised element-based limiters to remove spurious oscillations when projecting from discontinuous to continuous finite element spaces, which are necessary to make the whole transport scheme bounded.
3. When combined with standard limiters for the discontinuous Galerkin stage, the overall scheme remains locally bounded, addressing the previously unsolved problem of how to limit partially continuous finite element spaces that arise in the compatible finite element framework.

Our bounded transport scheme can also be used for continuous finite element methods, although other approaches are available that do not involve intermediate use of discontinuous Galerkin methods.

The rest of the paper is structured as follows. The problem is formulated in Section 2. In particular, more detail on the finite element spaces is provided in Section 2.1. The embedded discontinuous Galerkin schemes are introduced in Section 2.2; it is also shown that these schemes are stable if the underlying discontinuous Galerkin scheme is stable. The limiters are described in Section 2.3. In Section 3 we provide some numerical examples. Finally, in Section 4 we provide a summary and outlook.

2. Formulation

2.1. Finite element spaces

We begin by defining the partially continuous finite element spaces under consideration. In three dimensions, the element domain is constructed as the tensor product of a two-dimensional horizontal element domain (a triangle or a quadrilateral) and a one-dimensional vertical element domain (*i.e.*, an interval); we obtain triangular prism or hexahedral element domains aligned with the vertical direction. For a vertical slice geometry in two dimensions (frequently used in testcases during model development), the horizontal domain is also an interval, and we obtain quadrilateral elements aligned with the vertical direction.

To motivate the problem of transport schemes for a partially continuous finite element space, we first consider a compatible finite element scheme that uses a discontinuous finite element space for density. This is typically formed as the tensor product of the DG_k space in the horizontal (degree k polynomials on triangles or bi-k polynomials on quadrilaterals, allowing discontinuities between elements) and the DG_l space in the vertical. We consider the case where the same degree is chosen in horizontal and vertical, *i.e.* $k = l$, although there are no restrictions in the framework. We will denote this space as $DG_k \times DG_k$.

In the compatible finite element framework, the vertical velocity space is staggered in the vertical from the pressure space; the staggering is selected by requiring that the divergence (*i.e.*, the vertical derivative of the vertical velocity) maps from the vertical velocity space to the pressure space. This means that vertical velocity is stored as a field in $DG_k \times CG_{k+1}$ (where CG_{k+1} denotes degree $k + 1$ polynomials in each interval element, with C^0 continuity between elements). To avoid