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A semi-implicit spectral method for compressible convection of rotating and density-stratified flows in Cartesian geometry

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ABSTRACT

In this paper, we have described a 'stratified' semi-implicit spectral method to study compressible convection in Cartesian geometry. The full set of compressible hydrodynamic equations are solved in conservative forms. The numerical scheme is accurate and efficient, based on fast Fourier/sin/cos spectral transforms in the horizontal directions, Chebyshev spectral transform or second-order finite difference scheme in the vertical direction, and second order semi-implicit scheme in time marching of linear terms. We have checked the validity of both the fully pseudo-spectral scheme and the mixed finite-difference pseudo-spectral scheme by studying the onset of compressible convection. The difference of the critical Rayleigh number between our numerical result and the linear stability analysis is within two percent. Besides, we have computed the Mach numbers with different Rayleigh numbers in compressible convection. It shows good agreement with the numerical results of finite difference methods and finite volume method. This model has wide application in studying laminar and turbulent flow. Illustrative examples of application on horizontal convection, gravity waves, and long-lived vortex are given in this paper.

1. Introduction

Turbulent convection is a challenging problem in understanding astrophysical and geophysical flows. Numerical simulations provide a convenient tool to study turbulent convection in stars and planets. The simulations can be performed either in global geometry [21,22,13,30,5] or in local area geometry [33,8,28,37]. Both of them have advantages and disadvantages. The global simulation has the advantage of understanding the global structure of turbulent convection or dynamo effects in stars and planets [4,3,34,29,14]. Small scale structure of turbulent convection, on the other hand, can be hardly captured in global simulation because of the low spatial resolution. The local area simulation only focuses on small pieces of stars or planets, thus it has the disadvantage when studying large scale turbulent flows. However, it is still useful for the local area simulation to study the phenomena associated with higher degree of turbulence, such as solar granules or planetary vortex. Below we discuss the numerical aspect on the local area simulation.

Many of the astrophysical flows are deep and compressible, such as the solar convection or the atmosphere of giant planets. For simulations of deep convection zone, long time of numerical computation is usually required for thermal relaxation. It is well known that computational time step is usually constrained by the CFL (Courant–Friedrichs–Lewy) condition in explicit scheme [15]. To overcome this problem, four methods are frequently used. The first is taking anelastic approximation [22] or pseudo-incompressible approximation [18] to filter out sound wave; the second is integrating hydrodynamic

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equations with implicit method [8]; the third uses a technique to reduce sound speed. In the past three decades, several groups have developed numerical codes for local area simulations based on these methods [31,1,8,35,27]. On numerical scheme, most of them use finite central difference methods. The numerical schemes thus keep a second-order, fourth-order or sixth-order accuracy in spatial directions. The accuracy can be improved exponentially if spectral method is applied. Cattaneo et al. [7] have developed a spectral code for local area simulation, but the time integration is treated explicitly. Fan et al. [20] employ a semi-implicit mixed finite-difference pseudo-spectral code for local area simulation, but the acoustic wave is filtered out by anelastic approximation. In this paper, we keep the acoustic wave and introduce a semi-implicit spectral method for local area simulation of compressible convection.

In a previous paper, a semi-implicit stratified spectral model in spherical geometry has been developed by Chan et al. [11]. Later, the concept for stratified semi-implicit spectral model in Cartesian geometry has been introduced in Chan [10]. The semi-implicit stratified spectral method solves the full set of compressible hydrodynamic equations under the stratified approximation. The stratified approximation assumes the horizontal variations of hydrodynamic variables are small compared to the mean values on each layers. Therefore the higher order products of the hydrodynamic variables can be neglected from the perturbation equations. Compared to other methods, the stratified spectral method has advantages in the following aspects. First, the semi-implicit treatment of time integration is easy to implement thus allows us to use large time step without filtering out sound waves. Second, the spectral transform provides us with high accuracy on the spatial direction. Third, fast spectral transform makes it computationally efficient with moderate truncated spectral numbers. The stratified approximation keeps quadratic terms in the nonlinear terms, thus has an accuracy of the order Ma^2, where Ma is the Mach number defined by the ratio of local fluid speed to sound speed. Therefore the stratified approximation is mainly applicable to regions where the Mach number is small, otherwise the errors introduced by the stratified approximation are not negligible. In this paper, we describe the numerical scheme for the semi-implicit stratified spectral method in Cartesian geometry. Numerical tests are performed to validate the accuracy and efficiency of the method.

2. The model

2.1. Stratified approximation

The fundamental hydrodynamic equations for a compressible viscous flow in a rotation frame are:

$$\partial_t \rho = -\nabla \cdot \mathbf{M} \,, \tag{1}$$

$$\partial_t \mathbf{M} = -\nabla \cdot (\mathbf{M}\mathbf{M}/\rho) + \nabla \cdot \boldsymbol{\Sigma} - \nabla p + \rho \mathbf{g} - 2\boldsymbol{\Omega} \times \mathbf{M} \,, \tag{2}$$

$$\partial_t E = -\nabla \cdot [(E+p)\mathbf{M}/\rho - \mathbf{M} \cdot \boldsymbol{\Sigma}/\rho + \mathbf{f}] + \mathbf{M} \cdot \mathbf{g} \,. \tag{3}$$

The symbols have the following meanings: ρ is the density, $\mathbf{M} = \rho \mathbf{V}$ is the mass flux, $\mathbf{V}$ is the velocity, $\boldsymbol{\Sigma}$ is the viscous tensor, p is the pressure, $\boldsymbol{\Omega}$ is the angular velocity of the rotation frame, $\mathbf{g}$ is the gravitational acceleration, $E = e + \frac{1}{2}\rho V^2$ is the total energy density, $\mathbf{f}$ is the diffusive heat flux, e is the internal energy density, and $e = p/(\Gamma - 1)$ for an ideal gas, where Γ is the ratio of specific heats. The viscous stress term can be expressed by

$$\boldsymbol{\Sigma} = 2\mu \boldsymbol{\sigma} + \lambda(\nabla \cdot \mathbf{V})\mathbf{I} \,, \tag{4}$$

where $\sigma_{ij} = \frac{1}{2}(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i})$ is the strain rate tensor, μ is the dynamic viscosity, λ is the second viscosity, and $\mathbf{I}$ is the identity tensor.

For density-stratified flows, the horizontal fluctuations of the thermodynamic variables are usually smaller than their respective horizontal mean values. Therefore we decompose each thermodynamic variable into a time-dependent mean value and a fluctuating component that varies with space and time. The density, for example, is decomposed into the following form:

$$\rho(x, y, z, t) = \rho_0(z, t) + \rho_1(x, y, z, t) \,, \tag{5}$$

where ρ_0 is the time-dependent mean value on each 'stratified' layer, and ρ_1 is the fluctuating component.

Substituting the thermodynamic variables with the decomposed components, one can obtain the governing equations of the fluctuating variables. Higher order degree of nonlinearity appear in the momentum and energy equations because of the horizontal variation of density. The appearance of higher order nonlinear terms causes trouble in de-aliasing of the spectral transform. To overcome this problem, Chan et al. [11] have introduced a 'stratified approximation' to reduce the order of the nonlinearity to be quadratic. The 'stratified approximation' makes the following assumptions:

- the variation of density is ignored in the viscous terms and nonlinear advection terms in the momentum equation.
- the terms containing products of two or more horizontal variations of hydrodynamic variables are ignored in the energy equation.

By the 'stratified approximation', the relative errors introduced to the momentum and energy equations are on the order of $O(\mathrm{Ma}^2)$. The accuracy is confirmed both analytically and numerically by Chan et al. [11]. Stratified approximation and