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Optimizing the geometrical accuracy of curvilinear meshes

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ABSTRACT

This paper presents a method to generate valid high order meshes with optimized geometrical accuracy. The high order meshing procedure starts with a linear mesh, that is subsequently curved without taking care of the validity of the high order elements. An optimization procedure is then used to both untangle invalid elements and optimize the geometrical accuracy of the mesh. Standard measures of the distance between curves are considered to evaluate the geometrical accuracy in planar two-dimensional meshes, but they prove computationally too costly for optimization purposes. A fast estimate of the geometrical accuracy, based on Taylor expansions of the curves, is introduced. An unconstrained optimization procedure based on this estimate is shown to yield significant improvements in the geometrical accuracy of high order meshes, as measured by the standard Hausdorff distance between the geometrical model and the mesh. Several examples illustrate the beneficial impact of this method on CFD solutions, with a particular role of the enhanced mesh boundary smoothness.

1. Introduction

The development of high-order numerical technologies for engineering analysis has been underway for many years now. For example, Discontinuous Galerkin methods (DGM) have been thoroughly studied in the literature, initially in a theoretical context [1], and now from the application point of view [2,3]. Compared to standard second-order-accurate numerical schemes, high-order methods exhibit superior efficiency in problems with high resolution requirements, because they reach the required accuracy with much coarser grids.

However, many contributions have pointed out that the accuracy of these methods can be severely hampered by a too crude discretization of the geometry [4–6]. It is now widely accepted that linear geometrical discretizations may annihilate the benefits of high-order schemes in cases featuring curved geometries, that is, in most cases of engineering and scientific interest.

This problem has motivated the development of isogeometric analysis [7], in which both the mesh and the geometry are expressed in bases of Non-Uniform Rational B-Splines (NURBS) functions defined on patches. This allows geometrical models, that are usually described by NURBS in Computer Aided Design (CAD) engines, to be exactly accounted for in the simulation. However, mesh generation for isogeometric methods is still an open problem. The NURBS-enhanced Finite Element Method (NEFEM) [8] retains the polynomial approximation of classical Finite Elements, but makes uses of a NURBS

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representation of the mesh boundaries. To our knowledge, no specific procedure has been devised to prevent non-planar NURBS boundaries from deteriorating the interior of NEFEM meshes.

Another approach, that is independent of the numerical scheme, is to generate high-order meshes in which curvilinear elements are meant to provide sufficient geometrical accuracy on the boundary. Elements are then most often defined in a Lagrangian polynomial basis by a set of high-order nodes. Until now, efforts have mostly been targeted at ensuring the validity of the mesh. Indeed, the naive approach consisting in simply curving the boundaries of a linear mesh to match the geometry often results in tangled elements [9]. The curvature of the boundary must somehow be "propagated" into the domain for all elements to be valid. In the case of locally structured meshes, such a situation can be avoided by means of an efficient isoparametric technique [10]. For unstructured meshes, untangling procedures based on topological operations [11–13], mechanical analogies [14–16] or optimization procedures [9,17] have been proposed.

Although the improved representation of the geometry of the domain is the prime motivation for the use of high-order meshes, only few authors have explicitly taken into consideration the quality of the geometrical approximation. In the literature, the limited work dedicated to this topic has focused on placing adequately the high-order nodes when curving the mesh boundaries. Simple techniques include interpolating them between the first-order boundary nodes in the parametric space describing the corresponding CAD entity [11,17] or projecting them on the geometry from their location on the straight-sided element. More sophisticated procedures have also been proposed. In Ref. [14], the high-order nodes on boundary edges are interpolated in the physical space through a numerical procedure involving either the CAD parametrization (in the case of a mesh edge assigned to an edge of the geometrical model), or an approximation of the geodesic connecting the two first-order vertices (in the case of an edge located on a 3D surface). Nodes located within surface elements are obtained through a more sophisticated version of this procedure. Instead of interpolating, Sherwin and Peiró [18] use a mechanical analogy with chains of springs in equilibrium that yields the adequate node distribution along geometrical curves and geodesics for edge nodes. Two-dimensional nets of springs provide the appropriate distribution of surface element nodes.

This paper presents a method that makes it possible to build geometrically accurate curvilinear meshes. Unlike previous work reported in the literature, the representation of the model by the mesh is explicitly evaluated by measuring the geometrical error, i.e. the error in the approximation of the geometrical model by the high-order mesh boundary. For this purpose, we use either a classical definition of the *distance* between geometrical entities, or a novel fast estimate. The aim of the method is to minimize this geometrical error through the use of standard optimization algorithms. Although most of the paper deals with two-dimensional meshes, it is shown that the approach can easily be extended to three spatial dimensions.

Consider a model entity $\mathcal{C}$ and the mesh entity $\mathcal{C}_m$ that is meant to approximate $\mathcal{C}$. The first questions that arise are how to define a proper distance $d(\mathcal{C}, \mathcal{C}_m)$ between $\mathcal{C}$ and $\mathcal{C}_m$, and how to compute this distance efficiently. Two main definitions for such a distance have been proposed in the computational geometry literature, namely the Fréchet distance and the Hausdorff distance.

In this context of curvilinear meshing, distances $d(\mathcal{C}, \mathcal{C}_m)$ that are computed are usually small in comparison with the typical dimension of either $\mathcal{C}$ or $\mathcal{C}_m$. Consequently, $d(\mathcal{C}, \mathcal{C}_m)$ has to be computed with high accuracy. In this paper, we show that computing standard distances between the mesh and the geometrical model may be too expensive for the purpose of mesh optimization in practical cases. Alternative measures of distance are presented, that are both fast enough to compute and sufficiently accurate. An optimization procedure is then developed to drastically reduce the model-to-mesh distance while enforcing the mesh validity.

The paper is organized as follows. In Section 2, the problem of defining and computing a proper model-to-mesh distance is examined. The mesh optimization procedure is described in Section 3. Section 4 illustrates the method with examples, and the extension of the approach to three dimensions is presented in Section 5. Conclusions are drawn in Section 6.

2. Model-to-mesh distance

In order to optimize the representation of the geometrical model by the mesh boundaries, a proper method for quantifying the geometrical approximation error is required. As this error is to be computed for a possibly large number of boundary elements and used in an iterative optimization algorithm, the computational efficiency of the error evaluation algorithm is a crucial point, in addition to its accuracy.

In this section, we first consider a formal definition of the distance between planar curves, namely the Hausdorff distance, with the aim of measuring the geometrical error of a mesh edge approximating a model curve. We discuss the corresponding evaluation algorithms, and argue that even simplified forms of this distance is too costly for our purpose. Therefore, we introduce a novel estimate of the geometrical error that is computationally inexpensive and can easily be generalized to 3D problems, as discussed in Section 5.

2.1. Distance between curves

2.1.1. Setup

Consider the following planar parametric curve

$$\mathcal{C} \equiv \{\eta \in [\eta_0, \eta_p] \mapsto \boldsymbol{x}(\eta) \in R^2\}$$