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A curved boundary treatment for discontinuous Galerkin schemes solving time dependent problems

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ABSTRACT

For problems defined in a two-dimensional domain Ω with boundary conditions specified on a curve Γ, we consider discontinuous Galerkin (DG) schemes with high order polynomial basis functions on a geometry fitting triangular mesh. It is well known that directly imposing the given boundary conditions on a piecewise segment approximation boundary Γ_h will render any finite element method to be at most second order accurate. Unless the boundary conditions can be accurately transferred from Γ to Γ_h, in general curvilinear element method should be used to obtain high order accuracy. We discuss a simple boundary treatment which can be implemented as a modified DG scheme defined on triangles adjacent to Γ_h. Even though integration along the curve is still necessary, integrals over any curved element are avoided. If the domain Ω is convex, or if Ω is nonconvex and the true solutions can be smoothly extended to the exterior of Ω, the modified DG scheme is high order accurate. In these cases, numerical tests on first order and second order partial differential equations including hyperbolic systems and the scalar wave equation suggest that it is as accurate as the full curvilinear DG scheme.

1. Introduction

Consider solving a two-dimensional time dependent problem defined on a curved domain Ω with boundary conditions specified on a curve $\Gamma \subseteq \partial\Omega$. Assuming a geometry fitting triangular mesh is given, we focus on discontinuous Galerkin (DG) method with high order polynomial basis functions. For high order schemes defined on such a triangular mesh as illustrated in Fig. 1, boundary conditions on Γ_h as an approximation to Γ must be carefully treated to obtain optimal convergence rate. For instance, given homogeneous Dirichlet boundary conditions on Γ, any finite element method will be at most second order accurate with Dirichlet boundary conditions imposed on Γ_h [1,2]. Towards optimal convergence rate, a curved element near Γ can be used [3].

Even though the curvilinear element method via an isoparametric parametric approximation to Γ [4] is rather convenient to use for DG schemes [5–8], the computational and memory costs in curved elements will be increased due to integration on curved elements, especially when the boundary geometry is represented by very high order polynomials in high dimensions. Thus there is a strong motivation in studying more efficient alternatives to the full curvilinear DG methods.

One popular simple treatment to reduce computational cost of DG method on curved elements is to include the Jacobian determinant of the map from each curved element to a straight-sided reference element either in solution space or in test function space, e.g., [9,10]. Even though it may work well for a lot of problems in practice, such a nonpolynomial approx-

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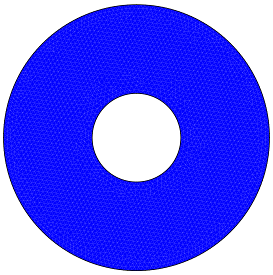
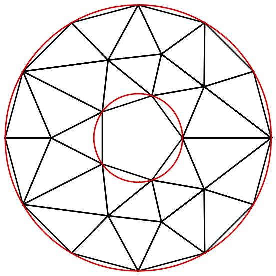


(a) The domain Ω.



(b) Γ consists of two red circles, approximated by piecewise segments Γ_h in the triangular mesh.

Fig. 1. An illustration of a geometry fitting triangular mesh on a curved domain. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

imation is not well understood in analysis. A low-storage curvilinear DG method was proposed and analyzed in [11,12], where the geometric factors were included in both solution and test function spaces with a provable convergence under a mild condition on the mesh. For tensor-product type elements, the mass matrix is lumpable on the curved elements, see [8].

For specific schemes and problems, it is possible to accurately transfer the boundary conditions from Γ to Γ_h so that high order accuracy can be obtained for DG on triangular meshes without curved elements. In [13], a simple approximation to curved solid wall boundary conditions for steady gas dynamics equations was discussed. An implicit transfer of boundary conditions was presented in [14,15] for the hybridizable DG method solving steady convection diffusion equations and the mesh does not need to be strictly geometry fitting in this method. For time dependent gas dynamics equations, an explicit transfer through an inverse Lax–Wendroff procedure was discussed for finite difference schemes in [16–19] yet the performance of this method applied to DG schemes is unclear.

Recently a simple curved interface treatment for DG scheme on triangles solving acoustic wave equations was presented in [20]. In this paper, we will extend the approach in [20] to treating curved boundaries. We derive a modification to the DG scheme defined on a boundary triangle. Even though the line integration along the curve Γ is still necessary, integrals over curved elements are avoided. By local truncation error analysis, such a modified DG scheme is high order accurate in convex domains. For nonconvex domains, it is also high order accurate if the equation and its smooth exact solution can be smoothly extended to the exterior of the domain. When the solution cannot be smoothly extended on nonconvex domains, the modified DG scheme is at most second order accurate however produces smaller errors than the DG scheme defined on triangles. On the other hand, a simple spectrum analysis suggest that this kind of modified DG scheme is unstable for arbitrary misfit between the boundary of a triangular mesh and the true curved boundary. Nonetheless, numerical tests suggest that such a scheme is stable on a reasonably coarse triangular mesh and finer ones.

The paper is organized as follows: we first discuss the main idea in Section 2 for first order equations. The same idea can be applied to other time dependent problems. As a demonstration, we discuss the second order wave equation in Section 3. For hyperbolic conservation laws, the local conservation is an important property. We discuss an additional step to enforce the local conservation in Section 4. Numerical tests are shown in Section 5. Section 6 consists of concluding remarks.

2. Time-dependent conservation laws

2.1. Preliminaries

Consider solving the following initial-boundary value problem on a two-dimensional curved domain Ω with $\Gamma \subseteq \partial\Omega$:

$$\begin{cases} u_t + \nabla \cdot \mathbf{F}(u) = 0, & \mathbf{x} \in \Omega, \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}), & \mathbf{x} \in \Omega, \\ u(\mathbf{x}, t) = b(\mathbf{x}, t), & \mathbf{x} \in \Gamma. \end{cases} \tag{1}$$

Suppose a triangular mesh T_h of the domain Ω fitting the boundary $\partial\Omega$ is given. For simplicity, we assume that the mesh fits the geometry in a way that Γ does not intersect any edge in T_h at more than two points. We also assume Γ does not pass more than two vertices of any triangle in T_h. These two assumptions are not essential for remaining discussion in this paper. Then for any boundary triangle K adjacent to the curve Γ, there are only two possibilities for the intersection between Γ and K. If Γ intersects K at only two vertices of K, we call it a convex case. Otherwise, Γ also intersects K at its interior, then we call it a concave case. Let e_K^i $(i = 1, 2, 3)$ be the three edges of the triangle K and e_k^1 be the one adjacent to the curve Γ. Let $\widetilde{e}_K^1$ be an isoparametric approximated representation of Γ, i.e., a high order polynomial interpolant of the curve Γ. We use $\widetilde{K}$ denote the curvilinear element bounded by e_K^2, e_K^3 and $\widetilde{e}_K^1$. Let C denote the difference between K and $\widetilde{K}$. See Fig. 2.