Journal of Computational Physics 308 (2016) 438–442

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

Hyperbolic systems of equations posed on erroneous curved domains

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ARTICLE INFO

Article history:
Received 21 September 2015
Received in revised form 18 December 2015
Accepted 23 December 2015
Available online 29 December 2015

Keywords:
Hyperbolic systems
Erroneous curved domains
Inaccurate data
Convergence rate

ABSTRACT

The effect of an inaccurate geometry description on the solution accuracy of a hyperbolic problem is discussed. The inaccurate geometry can for example come from an imperfect CAD system, a faulty mesh generator, bad measurements or simply a misconception.

We show that inaccurate geometry descriptions might lead to the wrong wave speeds, a misplacement of the boundary conditions, to the wrong boundary operator and a mismatch of boundary data.

The errors caused by an inaccurate geometry description may affect the solution more than the accuracy of the specific discretization techniques used. In extreme cases, the order of accuracy goes to zero. Numerical experiments corroborate the theoretical results.

1. Erroneous computational domain

Consider the following hyperbolic system of equations, in two space dimensions,

$$\begin{aligned} W_t + \hat{A}W_x + \hat{B}W_y &= 0, & (x, y) &\in \Omega, & t &\in (0, T), \\ LW &= g(x, y, t), & (x, y) &\in \delta\Omega, & t &\in (0, T), \\ W &= f(x, y), & (x, y) &\in \Omega, & t &= 0, \end{aligned} \tag{1}$$

in which the solution is represented by the vector $W = W(x, y, t)$. $\hat{A}$ and $\hat{B}$ are constant symmetric $M \times M$ matrices, Ω is the spatial domain with the boundary $\delta\Omega$. The boundary operator L is defined on $\delta\Omega$, $f(x, y) \in \mathbb{R}^M$ and $g(x, y, t) \in \mathbb{R}^M$ are the data in the problem.

Equation (1) is transformed to curvilinear coordinates (ξ, η) by $(V_\xi, V_\eta, V_t) = [J](V_x, V_y, V_t)^T$, where $[J]$ is the Jacobian matrix of the transformation. The transformed problem is

$$\begin{aligned} JW_t + AW_\xi + BW_\eta &= 0, & (\xi, \eta) &\in \Phi, & t &\in (0, T), \\ LW &= g(\xi, \eta, t), & (\xi, \eta) &\in \delta\Phi, & t &\in (0, T), \\ W &= f(\xi, \eta), & (\xi, \eta) &\in \Phi, & t &= 0, \end{aligned} \tag{2}$$

where $A = J\xi_x\hat{A} + J\xi_y\hat{B}$, $B = J\eta_x\hat{A} + J\eta_y\hat{B}$ and $J = x_\xi y_\eta - x_\eta y_\xi > 0$ is the determinant of $[J]$.

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http://dx.doi.org/10.1016/j.jcp.2015.12.048
0021-9991/

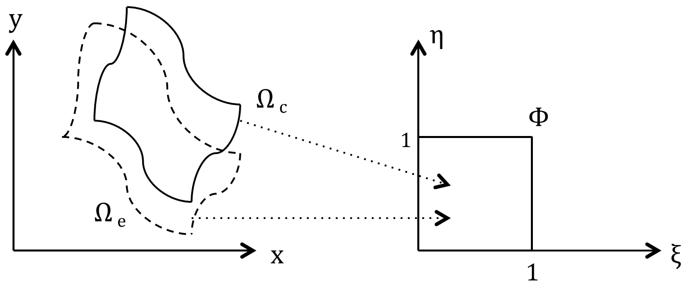

Fig. 1. A schematic of two different geometry descriptions both mapped to the unit square.

The energy method together with the metric identities and the use of the Green–Gauss theorem yields

$$\frac{d}{dt}\|W(\xi,\ \eta,\ t)\|_J^2 = -\oint_{\delta\Phi} W^T C\, W\, ds, \tag{3}$$

where the norm is defined by $\|V\|_J^2 = \int\int_\Phi V^T J\ V\, d\xi\, d\eta$. In (3), $C = (A,\ B)\cdot n = \alpha A + \beta B = X\Lambda X^T$, $n = (\alpha,\ \beta)$ is the unit normal vector pointing outward from Φ, X contains the eigenvectors as columns and Λ contains the eigenvalues on the main diagonal. For more details see [1].

The problem (2) is well-posed if we impose characteristic boundary conditions and the following energy rate is obtained

$$\frac{d}{dt}\|W(\xi,\eta,t)\|_J^2 + \oint_{\delta\Phi} W^T C^+ W\, ds = \oint_{\delta\Phi} g^T|\Lambda^-|g\, ds, \tag{4}$$

where Λ^- contains the negative eigenvalues of C and $C^+ = X\Lambda^+ C^T$.

1.1. Errors due to the wrong position of boundary

Consider two hyperbolic problems with solutions U and V posed on two nearby domains. The two domains, depicted in Fig. 1, are both mapped to the unit square. We consider Ω_c to be the correct domain and Ω_e to be the erroneous one (subscripts c and e denote correct and erroneous, respectively). The transformed equations become

$$J_c U_t + A_c U_\xi + B_c U_\eta = 0,\ \ J_e V_t + A_e V_\xi + B_e V_\eta = 0, \tag{5}$$

where the matrices are $A_i = [J\xi_x \hat{A} + J\xi_y \hat{B}]_i$ and $B_i = [J\eta_x \hat{A} + J\eta_y \hat{B}]_i$, for $i = c, e$. Our first result follows immediately. Since $A_c \neq A_e$, $B_c \neq B_e$ we realize that:

1) The wave speeds given by the eigenvalues of the matrices will differ, and that V will have the wrong wave speeds.

Next, we consider the results of the energy method given by (4) where $W \in \{U, V\}$ and $C_i = \alpha_i A_i + \beta_i B_i$ for $i = c, e$. A closer look at the boundary term C_i reveals that the normals as well as the eigenvalues of C_i are modified by the erroneous geometry. This paves the way for the second, third and fourth conclusions:

2) Error in the normal: This error is caused by imposing the wrong boundary operator due to an erroneous normal. Consider for example the solid wall no penetration condition $(u, v)\cdot n = \alpha u + \beta v = g = 0$ for the Euler equations. An erroneous normal will lead to the wrong boundary operator. This means that the boundary operator L in (2) is wrong while the data g is correct. See case 1 in Fig. 2.
3) Error in the position: Here the error is due to the misplacement of the boundary condition. Consider again the solid wall no penetration condition for the Euler equations, now with a correct boundary operator and data, imposed at the wrong position in space. In this case, the boundary operator L and the data g in (2) are both correct. See case 2 in Fig. 2.
4) Error in the data: Boundary conditions with data from Ω_c but imposed at Ω_e will also lead to inaccurate results. In this case, the data g in (2) is wrong while the boundary operator L can be either correct or wrong. See cases 2 and 3 in Fig. 2.

We can also have a combination of all the errors. These errors might be, and often are [4], more important than the order of accuracy of the specific discretization techniques used (which will be shown later in the numerical experiments section below).

To further discuss the effect of the wrong position of boundary conditions, consider (2) and two types of boundary data, correct and erroneous denoted by g_c and g_e, respectively. The correct and erroneous solutions are denoted by U and V,