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Fast difference schemes for solving high-dimensional time-fractional subdiffusion equations

Fanhai Zeng [a], Zhongqiang Zhang [b], George Em Karniadakis [a,*]

[a] Division of Applied Mathematics, Brown University, Providence RI, 02912, United States
[b] Department of Mathematical Sciences, Worcester Polytechnic Institute, Worcester, MA, 01609, United States

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ABSTRACT

In this paper, we focus on fast solvers with linearithmic complexity in space for high-dimensional time-fractional subdiffusion equations. Firstly, we present two alternating direction implicit (ADI) finite difference schemes for the two-dimensional time-fractional subdiffusion equation that are convergent of order $(1+\beta)$ in time, where β $(0<\beta<1)$ is the fractional order. Secondly, we develop two finite difference schemes which admit fast solvers without applying ADI techniques for two-dimensional time-fractional subdiffusion. Lastly, we extend these fast solvers to three-dimensional time-fractional subdiffusion. All the non-ADI difference methods are unconditionally stable and convergent with order two in time and order two or four in space. We also present several numerical experiments to verify the theoretical results.

1. Introduction

Anomalous diffusion, either subdiffusion or superdiffusion, is encountered in many diverse applications in science and engineering, see e.g. [1]. It is typically modeled through time-fractional derivatives, which give rise to great computational complexity caused by the non-local nature of the fractional operators. Numerical solution of the corresponding fractional differential equations (FDEs) is particularly problematic in high dimensions, so the majority of published works deals with one-dimensional problems whereas high dimensions are usually split following a classical alternating direction implicit (ADI) method. In this paper, we consider fast finite difference methods (FDMs) with linearithmic complexity for the following two-dimensional time-fractional subdiffusion equation, see e.g. [1–3]:

$$\begin{cases} {}_{C}D_{0,t}^{\beta}U=\mu\,\Delta U+f(x,y,t), \quad (x,y,t)\in\Omega\times(0,T],\, T>0,\\ U(x,y,0)=\phi_0(x,y), \quad x\in\Omega,\\ U(x,y,t)=0, \quad (x,y,t)\in\partial\Omega\times(0,T], \end{cases} \tag{1}$$

* Corresponding author.
E-mail addresses: fanhai_zeng@brown.edu (F. Zeng), zzhang7@wpi.edu (Z. Zhang), george_karniadakis@brown.edu (G.E. Karniadakis).

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and its three-dimensional counterpart, where $\Delta = \partial_x^2 + \partial_y^2$, $0 < \beta < 1$, $\mu > 0$, $\Omega = (x_L, x_R) \times (y_L, y_R)$, and ${}_C D_{0,t}^{\beta}$ is the βth-order Caputo derivative operator defined by [4]

$$ {}_C D_{0,t}^{\beta} U(\cdot,t) = D_{0,t}^{-(1-\beta)} [\partial_t U(\cdot,t)] = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} \partial_s U(\cdot,s)\, \mathrm{d}s, \tag{2} $$

in which $D_{0,t}^{-\beta}$ is the fractional integral operator defined by

$$ D_{0,t}^{-\beta} U(\cdot,t) = {}_{RL} D_{0,t}^{-\beta} U(\cdot,t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} U(\cdot,s)\, \mathrm{d}s, \quad \beta > 0. \tag{3} $$

Many numerical methods have been proposed to solve high-dimensional fractional partial differential equations (FPDEs) like (1): see e.g. [5–9] for space-fractional partial differential equations (PDEs), e.g. [3,10–24] for time-fractional PDEs, and e.g. [25–29] for time–space-fractional diffusion; see also the recent book [30] on a review of numerical methods for FDEs. Among all the numerical methods for high-dimensional FPDEs, only the ADI method is computationally efficient to be applied to solve the resulting linear systems with linear complexity, see e.g. [5–9,31–35]. However, there is a noticeable difference when ADI techniques are applied to time-fractional PDEs and integer-order PDEs: the convergence rate in time is degraded by the fractional order β, see e.g. [2,3,28,36–38], while for the integer-order PDEs, ADI techniques do not have such a limitation, see e.g. [39–41]. For ADI methods of the high-dimensional time-fractional subdiffusion equation of the type (1), the convergence rate in time is of order

- $\min\{q, 1+\beta\}$ (e.g., $q = 2-\beta$ in [2,3]) or
- $\min\{q, 2\beta\}$ (e.g., $q = 1$ in [37], $q = 2-\beta$ in [36,3], and $q = 2-\beta/2$ in [38]), or
- $\min\{q, \beta\}$ (e.g. $q = 2-\beta$ in [28]),

where q is the convergence rate of the time discretization methods applied together with the ADI method. Hence, when β is small, we achieve unsatisfactory accuracy in the existing ADI methods.

Two approaches have been proposed to improve the convergence rate of ADI methods. The first is to appropriately add some higher-order perturbation terms, see e.g. [2,40], while the other is to use the extrapolation method, see e.g. [31,42,43]. For these two approaches, no theoretical analysis has been presented to guarantee the stability.

In this paper, we use the first approach to increase the convergence rate of ADI methods, and we present two different ADI FDMs for (1). These two schemes are unconditionally stable and convergent with order $(1+\beta)$ in time and order two in space. However, the added perturbation terms may ruin the total accuracy, especially when β is small and/or $\partial_x^2 \partial_y^2 U(x,y,t)$ is large, see, e.g. [40] and Example 5.1 of Section 5.

We are then motivated to propose some non-ADI FDMs for the high-dimensional time-fractional subdiffusion equations (1) and (41) while we can still solve them with a low computational cost that is linearithmic with respect to the number of the grid points used in FDMs. Specifically, we present two fully non-ADI difference methods for (1) using the fractional linear multistep methods developed in [44] in time discretization and the standard central difference in physical space. Thanks to the special structure of the derived coefficient matrices, we can employ a fast eigen-solver with linearithmic complexity to solve the resulting linear systems. The fast solver allows us to solve the linear system directly with $O(N^2 \log(N))$ operations in space when we take N grid points in both x and y directions, instead of $O(N^3)$ operations for the direct solvers. We also prove that these two difference schemes are unconditionally stable with second-order accuracy both in time and space. In addition, we discuss how to achieve high-order convergence in physical space using compact finite difference schemes while we can still employ fast solvers without the ADI technique, see Section 3.2. Two compact non-ADI finite difference methods for (1) are proved to be both unconditionally stable and convergent with order two in time and four in space in Appendix A.

In Section 4, we show how the methodology presented in Section 3 can be extended to solve the three-dimensional time-fractional subdiffusion equation (41) with computational cost $O(N^3 \log(N))$ in physical space. The present methods are expected to work for d-dimensional time-fractional anomalous diffusion equations with $O(N^d \log(N))$ computational cost in physical space. There exist some fast solvers to solve FPDEs, such as [9,46], in which the ADI technique is used to convert the high-dimensional space-fractional PDEs into a series of one-dimensional ones, then the fast solver is applied. Here, we directly use the fast solver to solve the high-dimensional problems without using the ADI technique.

The rest of this paper is as follows. In Section 2, we derive two ADI FDMs for (1) and prove their stability and convergence rate. In Section 3, we develop two FDMs for (1) and employ a fast solver to solve the resulting linear system. We also consider two compact FDMs for (1). We present the stability and convergence rates of these schemes and leave the proofs the stability and convergence in Appendix A. We investigate the extension of the methodology to three-dimensional time-fractional subdiffusion in Section 4. In Section 5, we present numerical experiments to verify the theoretical results. We also present numerical comparisons between the present methods and the existing ones, both the ADI and non-ADI methods. In Section 6, we conclude and discuss our results.