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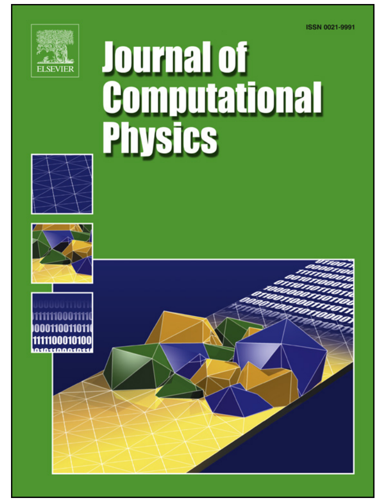

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EFFICIENT SPECTRAL-GALERKIN METHODS FOR FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

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ABSTRACT. Efficient Spectral-Galerkin algorithms are developed to solve multi-dimensional fractional elliptic equations with variable coefficients in conserved form as well as non-conserved form. These algorithms are extensions of the spectral-Galerkin algorithms for usual elliptic PDEs developed in [24]. More precisely, for separable FPDEs, we construct a direct method by using a matrix diagonalization approach, while for non-separable FPDEs, we employ an preconditioned BICGSTAB method with a suitable separable FPDE with constant-coefficients as preconditioner. The cost of these algorithms are of $O(N^{d+1})$ flops where d is the space dimension. We derive rigorous weighted error estimates which provide more precise convergence rate for problems with singularities at boundaries. We also present ample numerical results to validate the algorithms and error estimates.

1. INTRODUCTION

Fractional partial differential equations (FPDEs) appear in the investigation of transport dynamics in complex systems which are governed by the anomalous diffusion and non-exponential relaxation patterns [17]. Anomalous, or non-Fickian, dispersion has been an active area of research in the physics community since the introduction of continuous time random walks (CTRW) by Montroll and Weiss [18]. They have attracted considerable attention recently due to their ability to model certain processes which can not be adequately described by usual partial differential equations.

The classical 1-D diffusion equation

$$\partial_t p(x,t) - \partial_x[D(x)\partial_x p(x,t)] = 0, \quad x \in (a,b), \tag{1.1}$$

is derived from the conservation of mass

$$\partial_t p(x,t) + \partial_x F = 0, \tag{1.2}$$

and the Fick's first law

$$F = -D(x)\partial_x p(x,t), \tag{1.3}$$

where p is the density of the diffusing material at location x, F is the flux of the diffusing material, and $D(x)$ is the collective diffusion coefficient for density. Recently, scientists found that the classical Fick's law (1.3) is not adequate to describe anomalous diffusion which occurred in field and laboratory studies, such as transport of a solute in heterogeneous porous media. Instead, a fractional Fick's law is proposed [22]

$$F = -D(x)\Big(\frac{1}{2}(1+\kappa)\,{}_a\partial_x^\nu p(x,t) - \frac{1}{2}(1-\kappa)\,{}_x\partial_b^\nu p(x,t)\Big) \tag{1.4}$$

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