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On preservation of symmetry in r–z staggered Lagrangian schemes

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ABSTRACT

In the focus of this work are symmetry preservation, conservation of energy and volume, and other important properties of staggered Lagrangian hydrodynamic schemes in cylindrical (r–z) geometry. It is well known that on quadrilateral cells in r–z, preservation of spherical symmetry, perfect satisfaction of the Geometrical Conservation Law (GCL), and total energy conservation are incompatible even on conforming grids.
This paper suggests a novel staggered grid approach that preserves symmetry, conserves total energy by construction and tries to do its best by diminishing the GCL error to the order of entropy error. In particular, the forces from an existing volume consistent scheme are corrected so that spherical symmetry is preserved.
The incorporation of subcell pressure mechanism to reduce spurious grid deformations is described and the relation of the new scheme to popular area-weighted and control volume approaches studied.

1. Preface

It has been well-known that a spherically symmetric solution of the Euler equations can be maintained in cylindrical axi-symmetric (r–z) geometry. Doing this in the context of staggered grid Lagrangian schemes, while at the same time observing certain desirable theoretical properties, poses a challenge. In fact, exact symmetry preservation on polygonal cells is only possible for polar grids, and even then it is difficult to achieve on non-equiangular grids without compromising other properties such as momentum conservation [1]. We addressed what this challenge presents for the artificial viscous forces that are a necessary part of any staggered Lagrangian scheme in [2,3]. The properties of momentum conservation and internal energy increase in shocks are missing in the nevertheless effective tensor artificial viscous force of [4]. In [2,3] we presented a simple edge viscosity that satisfied those properties yet maintained symmetry when that was permitted by the grid, by means of an added correction to the radial component of the viscous force.

In this work we apply this idea of a radial correction to the pressure forces in the non-symmetry-preserving volume consistent scheme of [5], which we call GC for Geometric Conservation law (13). The symmetric version of this scheme, called GCS, is constructed together with the symmetry preserving (but not volume consistent) area weighted method of [6], which we call AW here. Then we present two classic test problems: the Sedov shock problem and the Coggeshall adiabatic compression problem. For the Sedov problem we show that GCS correctly tracks the shock, that is, the symmetry corrections are benign. The second problem is the Coggeshall adiabatic compression. For this we show that GCS does not reduce the

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accuracy of GC, in particular of the entropy, and here we have AW for reference. We also show that GCS has a subcell pressure [7] symmetry preserving capability that is applied to a perturbed grid for the Sedov data.

Finally, additional theoretical results are given on the symmetrization of the control volume (CV) scheme from [6], called CVS.

2. Notation and theory

2.1. Basics of a staggered grid Lagrangian scheme

In a popular notation the geometric objects in a Lagrangian discrete scheme are points indexed by $\{p\}$ and cells (or zones) indexed by $\{c\}$. Typically, cells are polygons and points are vertexes of the cells (nodes of the grid). For each point p, $c(p)$ is the set of cells that have p as a node (vertex), while $p(c)$ is the set of nodes of cell c. Each node is assigned a mass m_p and each cell is assigned a mass m_c. We assume there is a mass matrix with invariant elements m_{pc} such that

$$m_p = \sum_{c(p)} m_{pc}, \tag{1a}$$

$$m_c = \sum_{p(c)} m_{pc}, \tag{1b}$$

$$\frac{d}{dt} m_{pc} = 0. \tag{1c}$$

Dynamic objects include nodal velocities $\boldsymbol{U}_p$, nodal positions $\boldsymbol{X}_p$, cell volumes V_c which are explicit functions of nodal positions,

$$V_c = V(\boldsymbol{X}_p), \tag{2}$$

cell internal energies ε_c, and cell pressures P_c determined by an equation of state

$$P_c = P(V_c, \varepsilon_c). \tag{3}$$

The nodal dynamics are contained in the relations

$$\frac{d}{dt}\boldsymbol{X}_p = \boldsymbol{U}_p, \tag{4}$$

$$\boldsymbol{X}_p(t) = \boldsymbol{X}_p^n + (t - t^n)\, \boldsymbol{U}_p^{n+\frac{1}{2}}, \tag{5}$$

where

$$\boldsymbol{U}_p^{n+\frac{1}{2}} = \frac{1}{2}\left(\boldsymbol{U}_p^{n+1} + \boldsymbol{U}_p^n\right), \tag{6}$$

and

$$m_p\left(\boldsymbol{U}_p^{n+1} - \boldsymbol{U}_p^n\right) = \sum_{c(p)} P_c^{n+\frac{1}{2}} \boldsymbol{G}_{cp}, \tag{7}$$

where the sum is taken over all cells that share p as a vertex. In this work we calculate the time-centered pressure

$$P_c^{n+\frac{1}{2}} = \frac{1}{2}\left(P_c^{n+1} + P_c^n\right) \tag{8}$$

by an outer predictor–corrector type iteration on energy, where P_c^{n+1} is determined by the equation of state (3) applied to internal energy and volume at time t^{n+1}. Refer to section 6.4.2 of [2] for the complete algorithm, including details on inner and outer iterations, artificial viscosity, etc.

The matrix $\boldsymbol{G}$ consists of geometrical terms of the form

$$\boldsymbol{G}_{cp} = \int_{t^n}^{t^{n+1}} \boldsymbol{g}_{cp}(t)\, dt \tag{9}$$

with the integrands $\boldsymbol{g}_{cp}$ defined by the specific methods.

The energy dynamics are expressed in the system

$$m_c\left(\varepsilon_c^{n+1} - \varepsilon_c^n\right) = -P_c^{n+\frac{1}{2}} \sum_{p(c)} \boldsymbol{U}_p^{n+\frac{1}{2}} \cdot \boldsymbol{G}_{cp}, \tag{10}$$