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Existence and stability in the virtual interpolation point method for the Stokes equations

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ABSTRACT

In this paper, we propose a novel virtual interpolation point (VIP) method formulating discrete Stokes equations. We have formed virtual staggered structure for velocity and pressure from the actual computation node set. The VIP method by a point collocation scheme is well suited for meshfree scheme because the approximation comes from smooth kernels and kernels can be differentiated directly. This paper highlights our contribution to a stable flow computation without explicit structure of staggered grid. Our method eliminates the need to construct explicit staggered grid. Instead, virtual interpolation nodes play key roles in discretizing the conservative quantities of the Stokes equations. We have proved the inf–sup condition for VIP method with virtual structure of staggered grid and thus the existence and stability of discrete solutions follow.

1. Introduction

Despite the fact that there have been lots of schemes to solve flow problems, for example, the incompressible Navier–Stokes flow, the Euler flow which is compressible or incompressible, and the compressible Navier–Stokes flow, the issues on stability, efficiency, and accuracy take place frequently as the complexity of the problem increases. The finite volume method (FVM) which has long history uses the staggered grid for velocity and pressure for the purpose of stability.

We are concerned with the existence and stability of the numerical approximation of the stationary incompressible Stokes equations by VIP method derived from meshfree scheme. In the finite element method (FEM), there are extensive works for the inf–sup stability like Babuška [1], Brezzi [2], and Girault and Raviart [3]. Motivated by FEM, we form virtual interpolation node grid for velocity and pressure to exploit the inf–sup stability of staggered structure and then from the interpolation using collocation we prove the existence of discrete solution.

VIP method has great advantages in geometrical complexity and numerical algorithm compared with FEM and FVM because pointwise differentiation in formulation is adopted, single node set for both velocity and pressure is used and regular virtual interpolation node set can be constructed easily. We think our idea combining the virtual staggered structure and interpolation is very powerful to solve many difficult fluid problems.

The meshfree scheme has been successfully applied to various problems in fluid as appeared in Choe et al. [4], Park et al. [5], and Park [6]. One of the significant features of meshfree scheme is the versatility of reproducing kernel like complete local generation of polynomials. In this paper, we adopt point collocation method to formulate the discrete Stokes equations.

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The point collocation method is well suited to meshfree scheme since the approximation comes from smooth kernel and we can differentiate directly the kernels. For more details of basis function (shape function), Ψ, we refer to Liu et al. [7]. We include several numerical results to verify our theory.

In mathematics, we consider two dimensional stationary Stokes problem with periodic boundary condition,

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f}, \\ \operatorname{div}\mathbf{u} &= 0, \end{aligned} \tag{1}$$

in the periodic domain $\Omega = \mathbb{R}^2/\mathbb{Z}^2$, where $\mathbf{u}$ is velocity, p is pressure, and $\mathbf{f}$ is external force. From Helmholtz–Weyl decomposition, when $\mathbf{f} \in L^2(\Omega)$, we have that $\mathbf{f} = \nabla a + \mathbf{d}$, $\operatorname{div}\mathbf{d} = 0$ weakly in $L^2(\Omega)$. Therefore, by merging ∇a to pressure, we can assume $\mathbf{f}$ is solenoidal in (1). Furthermore taking divergence we find the pressure p is harmonic, although we do not need harmonicity in formulation, namely,

$$\Delta p = 0.$$

Let $X = H^1_{per}(\Omega) = \{\mathbf{u} : \mathbf{u}$ is periodic, $\int_\Omega \mathbf{u} d\mathbf{x} = 0$ and $\|\mathbf{u}\|^2_X = \int_\Omega |\nabla \mathbf{u}|^2 d\mathbf{x} < \infty\}$ and $M = L^2_{per}(\Omega) = \{q : q$ is periodic and $\int_\Omega |q|^2 d\mathbf{x} < \infty\}$. By the saddle point argument for the function space $X \times M$, the existence of the solution to the Stokes equations follows from the inf–sup condition as long as $\mathbf{f} \in H^{-1}_{per}(\Omega)$.

Definition 1.1. $X \times M$ satisfies inf–sup condition for a bilinear form b if there is a positive constant $\mu > 0$ such that

$$\inf_{p \in M \setminus \{0\}} \sup_{\mathbf{u} \in X} \frac{b(\mathbf{u}, p)}{\|\mathbf{u}\|_X \|p\|_M} \geq \mu > 0.$$

Proposition 1.2. *(See [8].) Suppose that $X \times M$ satisfies inf–sup condition for a bilinear form b. Given $\mathbf{f} \in X'$, there is a pair $(\mathbf{u}, p) \in X \times M$ such that*

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= \langle \mathbf{f}, \mathbf{v} \rangle \qquad \forall \mathbf{v} \in X, \\ b(\mathbf{u}, q) &= 0 \qquad \forall q \in M, \end{aligned}$$

where $a(\mathbf{u}, \mathbf{v}) = \int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{v} d\mathbf{x}$ and $b(\mathbf{u}, q) = \int_\Omega \operatorname{div}\mathbf{u} q d\mathbf{x}$. Moreover $(\mathbf{u}, p)$ satisfies

$$\|\mathbf{u}\|_X + \|p\|_M \leq C \|\mathbf{f}\|_{X'},$$

for a constant $C > 0$.

We discretize the incompressible Stokes equations by meshfree scheme. Then by the inf–sup condition for discrete version in Theorem 3.2, we prove existence and stability of VIP method. The most important contribution in this paper is the single node scheme for both velocity and pressure by VIP method. As a natural consequence, the computation becomes very efficient and stable and is very robust to geometrical complexity. Although the approximation node set may not have any structural condition, the numerical stability follows from the fact that VIP method compromises the usual staggered grid and the fact that any discrete vector can be reproduced by meshfree scheme. Since the collocation method requires the pointwise evaluation of the second derivatives at each node, we need higher regularity on the external force $\mathbf{f} \in C^\alpha(\Omega)$ to get approximation error. Theorems 3.2 and 3.5 are our main theorems for existence and stability.

To validate VIP method, we conduct several numerical simulations.

2. Formulation of VIP method

First, we introduce the meshfree method in view of moving least square by general setting and then consider the periodic domain. We let $\Omega \subset \mathbb{R}^n$ and u be a bounded C^∞ function. We consider the set of polynomials of degree less than or equal to m; for $\mathbf{x} = (x_1, x_2, \cdots, x_n)$,

$$\mathbb{P}_m = \operatorname{span}\{x_1^{\alpha_1} \cdots x_n^{\alpha_n} : |\alpha| = \alpha_1 + \cdots + \alpha_n \leq m\}. \tag{2}$$

Let $\mathbf{p}_m$ be a vector of polynomials, the components of $\mathbf{p}_m$ consist of all elements in $\mathbb{P}_m$, and elements are placed with multi-index ordering. For example, if we choose dimension $n = 2$ and polynomial degree $m = 2$, then $\mathbf{p}_m = (1, x, y, x^2, xy, y^2)$. We want to find the optimal local approximation of $u(\mathbf{x})$ in a ball $B_\rho(\mathbf{x})$ for a positive dilation parameter ρ. In details, we decide the coefficient vector $\mathbf{a}(\bar{\mathbf{x}})$ by the moving least square method. We introduce an error residual functional

$$J(\mathbf{a}(\bar{\mathbf{x}})) = \int_\Omega \left| u(\mathbf{x}) - \mathbf{p}_m\left(\frac{\mathbf{x} - \bar{\mathbf{x}}}{\rho}\right) \cdot \mathbf{a}(\bar{\mathbf{x}}) \right|^2 \frac{1}{\rho^n} \Phi\left(\frac{\mathbf{x} - \bar{\mathbf{x}}}{\rho}\right) d\mathbf{x},$$