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# Explicit filtering and exact reconstruction of the sub-filter stresses in large eddy simulation

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ABSTRACT

Explicit filtering has the effect of reducing numerical or aliasing errors near the grid scale in large eddy simulation (LES). We use a differential filter, namely the inverse Helmholtz operator, which is readily applied to unstructured meshes. The filter is invertible, which allows the sub-filter scale (SFS) stresses to be exactly reconstructed in terms of the filtered solution. Unlike eddy viscosity models, the method of filtering and reconstruction avoids making any physical assumptions and is therefore valid in any flow regime. The sub-grid scale (SGS) stresses are not recoverable by reconstruction, but the second-order finite element method used here is an adequate source of numerical dissipation in lieu of an SGS model. Results for incompressible turbulent channel flow at $Re_\tau = 180$ are presented which show that explicit filtering and exact SFS reconstruction is a significant improvement over the standard LES approach of implicit filtering and eddy-viscosity SGS modelling.



## 1. Introduction

LES is a widely-used technique for high-fidelity numerical simulation of turbulent flows. By resolving large-scale flow features and modelling the effects of smaller scales, detailed studies of flows can be carried out without the extreme computational cost of direct numerical simulation (DNS). The separation of large and small scales is achieved by filtering the Navier–Stokes equations with a spatial filter operator, resulting in the appearance of a closure term representing the divergence of the unresolved stresses. This closure term is unknown and must be modelled somehow using information from the resolved scales. Considerable efforts have been expended on the 'closure problem' as surveyed by, for example, [1–3]. This paper is not concerned with model development or assessment, but with the reduction of numerical errors in solutions of the filtered Navier–Stokes equations.

Before proceeding we must distinguish between two types of filtering in LES. Firstly, the numerical discretisation and mesh truncate the unknown exact solution, acting like an implicit filter with a cutoff lengthscale (denoted $\triangle$) determined by the mesh resolution, resulting in a closure term representing scales below the grid scale: i.e., the sub-grid scale (SGS) stresses, denoted $\tau_{SGS}$. Generally the exact form of the implicit filter operator is not known (although some have explored this topic [4–6]). Secondly, a filter operator with a cutoff lengthscale or 'width' $\widetilde{\triangle} > \triangle$ may be explicitly applied to the discrete equations. The filter is usually an integral operator with a specified kernel such as a Gaussian function. Then an additional closure term appears, representing grid-resolvable scales below the filter width $\widetilde{\triangle}$. These are called the sub-filter-scale (SFS) stresses, denoted $\tau_{SFS}$ (resolved SFS (RSFS) is sometimes used in the literature [7]).

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It may seem that additional modelling difficulties are introduced by explicit filtering, since both $\tau_{SFS}$ and $\tau_{SGS}$ have to be modelled. However, it is possible to exactly reconstruct $\tau_{SFS}$ in terms of the well-resolved wavenumbers by means of an invertible filter. Germano [8] proposed using a differential filter, namely the inverse Helmholtz operator, for which an exact inverse (defiltering) operator exists, enabling the SFS stresses to be exactly reconstructed. Exactly invertible filters were also investigated by Carati et al. [9], who proved that for any symmetric filter in one dimension, an exact expression could be derived as an infinite series expansion. They extended the idea to three dimensions by means of tensor products. Jameson also derived an exact expression for the SFS stresses in the compressible Navier–Stokes equations by using a generic invertible filter in the definition of the Favre-averaged variables [10]. Closely related methods are the approximate deconvolution (AD) [11] and velocity estimation techniques [12], whereby the filter is approximately inverted to generate a closure model for LES. These approaches have been successfully applied in a range of test cases [5,13,14].

A criticism leveled at exact reconstruction is that nothing is changed by explicitly filtering and adding back in exactly what was removed [5,9,15,16]. Certainly, this is true for the continuous system of PDEs. However, the discretised system has a distinct difference related to resolution of the turbulence spectrum. Typical numerical schemes exhibit increased dissipative and dispersive errors at the smaller scales close to $\triangle$. Explicit filtering is useful as a means of reducing the errors [17,18]. If the explicit filter width/cutoff frequency is chosen such that poorly resolved small scales/high frequencies (dependent on the numerical scheme [19]) are filtered out, then the remaining spectrum contains only well-resolved wavenumbers. The reconstructed SFS stress term only contains well-resolved wavenumbers [5,11]. Thus the procedure is akin to dealiasing and other filter-based stabilisation techniques. Furthermore, grid-converged LES solutions can be obtained by holding $\widetilde{\triangle}$ constant and refining $\triangle$, such that the modelling error is constant while the discretisation error is converged [20,21].

The inverse Helmholtz filter has been used by several researchers for LES [21–25]. Implementation of the filter in complex domains and on unstructured meshes is trivial because one simply needs to discretise and solve an elliptic equation using the existing numerical architecture. By contrast, integral filters, which essentially perform a weighted average over a patch of cells surrounding a node, are somewhat complicated to construct on unstructured meshes and in parallelised codes. Methods for constructing integral filters on unstructured meshes rely on complicated logic [26,27].

Exact reconstruction of the SFS stresses does not account for the SGS stresses, which are not directly recoverable from the resolved scales [9,28]. Additional modelling is required, for example with an eddy viscosity model [9,16]. We do not employ a model, but rely on numerical dissipation contributed by the second-order accurate stabilised finite element method used to discretise the equations. Approximation of the SGS stresses by numerical dissipation is broadly known as implicit LES (ILES). There is no guarantee that numerical dissipation represents the SGS dynamics [28], although it is an attractive approach for its simplicity and avoidance of the eddy-viscosity assumption.

In this paper, we apply explicit filtering and exact SFS reconstruction to LES of incompressible turbulent channel flow at $Re_\tau = 180$. Results show that the method leads to significantly improved predictions of bulk quantities (including skin friction) and first- and second-order flow quantities compared to the dynamic Smagorinsky model, no model, and explicit filtering only. The method does not rely on restrictive or inaccurate assumptions, such as the Boussinesq hypothesis or having the filter width sufficiently far down the inertial range, so it can be applied to any flow. It is formulated for arbitrary unstructured meshes and has the potential to improve the accuracy of LES of complex flows.

This paper is organised as follows. Section 2 introduces the filtering and reconstruction techniques. The exact reconstruction is derived and the commutation error is examined. Boundary conditions and similarities to other models are described. In Section 3, the stabilised finite element method is outlined and the numerical discretisation of the filter is presented. Section 4 presents the results of the turbulent channel flow validation case. Finally, Section 5 contains a discussion of the findings, conclusions and future work.

## 2. Explicit filtering and exact reconstruction of the SFS stresses

### 2.1. Filtered Navier–Stokes equations

We consider the incompressible Navier–Stokes equations for velocity $\mathbf{u} = \{u, v, w\}$ and pressure $p$ in a domain $\Omega \in \mathcal{R}^3$ with a boundary $\Gamma$, at a sufficiently high Reynolds number $Re$ that a turbulent energy cascade is present in the flow. Let us assume that the equations are discretised on a mesh that does not resolve the smallest scales of motion in the turbulent cascade, i.e. the computation is an LES rather than a DNS. The smallest scales are implicitly filtered out, and we denote the *implicitly filtered* solution by $\bar{\mathbf{u}}$. The implicitly filtered Navier–Stokes equations are

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\overline{\mathbf{u}\mathbf{u}^T}) + \frac{1}{\rho}\nabla \bar{p} - \nu \nabla^2 \bar{\mathbf{u}} = 0,$$
$$\nabla \cdot \bar{\mathbf{u}} = 0. \tag{1}$$

The unknown closure term $\overline{\mathbf{u}\mathbf{u}^T}$ is replaced by

$$\overline{\mathbf{u}\mathbf{u}^T} = \bar{\mathbf{u}}\,\bar{\mathbf{u}}^T + \tau_{SGS}, \tag{2}$$