Journal of Computational Physics 306 (2016) 167–198

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

# A Fourier penalty method for solving the time-dependent Maxwell's equations in domains with curved boundaries

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ARTICLE INFO

*Article history:*
Received 7 April 2015
Received in revised form 14 November 2015
Accepted 17 November 2015
Available online 1 December 2015

*Keywords:*
Active penalty method
Sharp mask function
Fourier methods
Maxwell equations
Fourier continuation

ABSTRACT

We present a high order, Fourier penalty method for the Maxwell's equations in the vicinity of perfect electric conductor boundary conditions. The approach relies on extending the smooth non-periodic domain of the equations to a periodic domain by removing the exact boundary conditions and introducing an analytic forcing term in the extended domain. The forcing, or penalty term is chosen to systematically enforce the boundary conditions to high order in the penalty parameter, which then allows for higher order numerical methods. We present an efficient numerical method for constructing the penalty term, and discretize the resulting equations using a Fourier spectral method. We demonstrate convergence orders of up to 3.5 for the one-dimensional Maxwell's equations, and show that the numerical method does not suffer from dispersion (or pollution) errors. We also illustrate the approach in two dimensions and demonstrate convergence orders of 2.5 for transverse magnetic modes and 1.5 for the transverse electric modes. We conclude the paper with numerous test cases in dimensions two and three including waves traveling in a bent waveguide, and scattering off of a windmill-like geometry.



## 1. Introduction

Pseudospectral and Fourier based methods [36] provide a popular solution approach for problems involving periodic boundary conditions. Unfortunately, pseudospectral methods which exploit the Fourier transform do not extend easily to domains with curved boundaries. One approach for solving partial differential equations (PDEs) on domains with curved boundaries is to relax the boundary condition by introducing a forcing, or penalty term to approximately enforce the correct boundary values. Such an approach has successfully been developed for a variety of problems in fluid dynamics [3,5,6,33] as well as computations involving turbulent flows [20]. Other more recent applications include using penalty equations in ocean modeling [32], plasma physics [4], magneto-hydrodynamics [28], and scalar advection with moving obstacles [19]. One significant drawback with such volume based penalty methods is the introduction of analytic errors in the penalized PDE. The resulting analytic error not only limits the accuracy of any numerical method, but also degrades the smoothness of the underlying solution. As a result of the reduced regularity in the penalized solution, the Fourier spectral methods typically require additional filtering steps [21].

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http://dx.doi.org/10.1016/j.jcp.2015.11.031

In recent work [34], a new modified penalty term was introduced to alleviate the analytic error due to the standard volume penalty method. The approach was examined for the heat and Poisson equations to obtain a third order Fourier-based method. The method was then extended to the Navier–Stokes equations to obtain a second order Fourier scheme.

The focus in the current paper is on hyperbolic wave equations with an emphasis on Maxwell's equations. Specifically, we focus on the time-dependent Maxwell's equations in free space in the presence of perfect electric conductors (PEC). Perfect conductors are idealized materials that easily conduct electricity and are accompanied by corresponding boundary conditions. Mathematically, PEC boundary conditions are modeled by assuming the electric field is normal to the boundary of the conducting material. Such a condition may then be converted into an appropriate Dirichlet boundary condition on the underlying PDE.

In contrast to previous work [34] which focused primarily on elliptic and parabolic equations, here a modified approach must be applied for hyperbolic systems. Specifically, the penalty term cannot be directly applied to a second order wave equation as it will introduce spurious oscillations in time, but rather must be introduced into the first order system so as to dampen solutions. Even with the suitable introduction of a penalty term to a hyperbolic system, the presence of analytic errors can significantly limit the accuracy of a numerical method. For instance we demonstrate that a conventional volume penalty method will converge at a rate of 0.5, namely the error scales as $O(\Delta x^{1/2})$ where $\Delta x$ is the grid spacing of the scheme. Recent work by [7] suggests that an alternative penalization may yield first order methods, while other work [10] shows second order convergence rates for a class of hyperbolic systems with Neumann boundary conditions. Finally, a similar in spirit approach [12,13,18,29], where an additional penalty term is prescribed to connect subdomains, or to enforce boundary conditions was developed to obtain provably stable numerical schemes. Although that method is currently limited to low order for boundary conditions [29], it is hopeful that future work may lead to the development of provably higher order penalty methods.

Another successful approach for solving wave problems with Fourier series is through Fourier extension methods [11, 26]. The methods have been very successful at obtaining highly accurate solutions for wave problems that do not have a divergence constraint. In particular, the Fourier method is combined with an iterative (alternate direction iteration) method to solve a sequence of elliptic problems as a means to evolve wave equations. The methods we propose in this paper differ as they may be discretized with an explicit in time method and therefore do not require solving an elliptic problem at each time iteration.

We emphasize that our approach is a single domain pseudospectral in space finite difference in time method. Previous single domain pseudospectral time-domain (PSTD) approaches [22–25] cannot handle curved geometries with PEC boundary conditions. Our approach can be thought of as a new way to extend the single domain PSTD method to domains with curved geometries. In addition, our approach preserves the use of the fast Fourier transform (FFT) and does not suffer from dispersion errors. In subsequent developments of the PSTD method [16] the FFT is no longer used, multiple domains must be introduced, and accuracy is lost due to subdomain coupling.

In methods such as the immersed boundary or standard penalty method, the extended solution is no longer smooth. The lack of smoothness then limits the convergence rate. As part of our approach, we ensure that the forcing creates an extension that is smooth in a precise sense. We demonstrate that with an appropriate modification and introduction of an active penalty term, one may achieve systematically higher order methods. Specifically, we show that for problems in one dimension, one may achieve convergence rates of up to 3.5 (the limitation currently due to time stepping), while in dimension two, one may obtain rates of 1.5 for transverse electric (TE) modes and 2.5 for transverse magnetic (TM) modes.

In the first half of the paper we introduce the Maxwell's equations with PEC boundary conditions, along with the formulation of the active penalty term. We also describe the analytic construction of the penalty term for TE and TM modes in dimension two. We then examine the analytic error in the penalty parameter for scattering of a TM mode off of a PEC wall. The second half of the paper focuses on the numerical implementation of solving the penalized Maxwell's equations using a Fourier pseudospectral approach. Specifically, we provide details on how to numerically discretize the equations in both space and time using equispaced grids and Fourier series. We then go on to outline details of stability studies in dimensions one and two and illustrate how the penalty term can be combined with PMLs to provide full time-dependent simulations of waves with PEC and radiating boundary conditions on periodic domains. In addition, we validate the approach by performing several numerical studies. Specifically, we show that in dimension one, the Fourier spectral method does not suffer from pollution (numerical dispersion) errors. We perform convergence studies in both one and two dimensions, showing global convergence rates of up to 3.5 in dimension one, 1.5 for TE modes in dimension two and 2.5 for TM modes in dimension two. Lastly, we illustrate the utility of the approach on some problems involving windmill shaped and waveguide geometries and demonstrate the natural extension to three dimensions.

## 2. Basic approach

In this paper we develop numerical Fourier methods for solving the time-dependent boundary value problem for Maxwell's equations. Specifically, we focus on solving Maxwell's equations for isotropic space in the vicinity of PEC. We denote the region of isotropic space by $\Omega_0 \subset \Omega = [0, D]^d$, for $d = 1, 2, 3$ where $[0, D]^d$ is the $d$-dimensional cube with periodic boundary conditions, and the boundary $\Gamma = \partial\Omega_0$. The Maxwell's equations then take the form