Journal of Computational Physics 306 (2016) 320–342

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

The Cauchy–Lagrangian method for numerical analysis of Euler flow

O. Podvigina [a], V. Zheligovsky [a,*], U. Frisch [b]

[a] Institute of Earthquake Prediction Theory and Mathematical Geophysics, Russian Acad. Sci., 84/32 Profsoyuznaya St., 117997 Moscow, Russian Federation
[b] Lab. Lagrange, UCA, OCA, CNRS, CS 34229, 06304 Nice Cedex 4, France

ARTICLE INFO

Article history:
Received 17 May 2015
Received in revised form 10 November 2015
Accepted 21 November 2015
Available online 2 December 2015

Keywords:
Euler equation
Lagrangian coordinates
Cauchy invariants
Semi-Lagrangian methods
High-order temporal schemes
High-precision schemes

ABSTRACT

A novel semi-Lagrangian method is introduced to solve numerically the Euler equation for ideal incompressible flow in arbitrary space dimension. It exploits the time-analyticity of fluid particle trajectories and requires, in principle, only limited spatial smoothness of the initial data. Efficient generation of high-order time-Taylor coefficients is made possible by a recurrence relation that follows from the Cauchy invariants formulation of the Euler equation (Zheligovsky and Frisch, 2014 [44]). Truncated time-Taylor series of very high order allow the use of time steps vastly exceeding the Courant–Friedrichs–Lewy limit, without compromising the accuracy of the solution. Tests performed on the two-dimensional Euler equation indicate that the Cauchy–Lagrangian method is more – and occasionally much more – efficient and less prone to instability than Eulerian Runge–Kutta methods, and less prone to rapid growth of rounding errors than the high-order Eulerian time-Taylor algorithm. We also develop tools of analysis adapted to the Cauchy–Lagrangian method, such as the monitoring of the radius of convergence of the time-Taylor series. Certain other fluid equations can be handled similarly.

1. Introduction

As is well known, fluid flow can be characterised in terms of the current positions of fluid particles (Eulerian coordinates), or in terms of their initial positions (Lagrangian coordinates). For ideal (inviscid) incompressible fluid both formulations were introduced in the 18th century [15,28]. From both a theoretical and a numerical point of view, the Eulerian formulation seems to have a significant edge because it gives an explicit quadratic expression for the time-derivative of the velocity; the Lagrangian formulation has even been qualified “an agony”, because of its complexity [36].

However, the Eulerian time-stepping methods have one serious well-known drawback, the Courant–Friedrichs–Lewy (CFL) condition [12], which constrains the time step to be less than a dimensionless constant multiplied by the time needed to sweep across the spatial mesh at the maximum flow speed. As a consequence, the complexity of computations with N collocation points in each spatial direction is roughly $O(N^4)$ in three dimensions. Hence, progress in high-resolution numerical simulation is creepingly slow, in spite of Moore's law. (As we will see in Section 5, this is not the only drawback of Eulerian schemes.)

* Corresponding author.
E-mail addresses: olgap@mitp.ru (O. Podvigina), vlad@mitp.ru (V. Zheligovsky), uriel@oca.eu (U. Frisch).

http://dx.doi.org/10.1016/j.jcp.2015.11.045
0021-9991/

Foremost because of the CFL constraint, which may conflict, for example, with the desire to quickly produce well-resolved numerical weather forecast, there has been a strong incentive to develop *semi-Lagrangian* (SL) schemes, in which some form of Lagrangian integration – with time steps not constrained by CFL – alternates with a reversion to an Eulerian grid (see, e.g., [40] and references therein). The SL algorithms used so far, e.g., in geophysical fluid dynamics, engineering, mechanics and plasma physics, were designed for situations where satisfactory results can be obtained with rather low-order temporal schemes. They are thus not appropriate for numerical experimentation on delicate questions, such as the issue of blow-up in three-dimensional flow (see, e.g., [22]).

We propose a new SL algorithm, which we call the Cauchy–Lagrangian algorithm. It relies on Cauchy's Lagrangian formulation of the equations of ideal incompressible flow [9,20] and on recent results about the time-analyticity of Lagrangian trajectories [21,44]. The Cauchy–Lagrangian algorithm is particularly well-suited for problems where high precision is a prerequisite, and it is actually superior to Eulerian schemes.

Cauchy's 1815 Lagrangian equations are formulated in terms of the Lagrangian map from Lagrangian to Eulerian coordinates, $\boldsymbol{a} \to \boldsymbol{x} = \boldsymbol{x}(\boldsymbol{a}, t)$, defined as the solution of the ordinary differential equation for fluid particle trajectory, $\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x}, t)$, with the initial condition $\boldsymbol{x}(\boldsymbol{a}, 0) = \boldsymbol{a}$. Here, the dot denotes a (Lagrangian) time derivative. The three-dimensional Cauchy invariants equations are (see Section 2 for a simplified derivation):

$$\sum_{k=1}^{3} \nabla^{\mathrm{L}} \dot{x}_k \times \nabla^{\mathrm{L}} x_k = \boldsymbol{\omega}^{(\mathrm{init})}, \qquad \det(\nabla^{\mathrm{L}} \boldsymbol{x}) = 1, \tag{1}$$

where $\boldsymbol{\omega}^{(\mathrm{init})} = \nabla^{\mathrm{L}} \times \boldsymbol{v}^{(\mathrm{init})}$ denotes the initial vorticity and ∇^{L} is the Lagrangian gradient, i.e., the space derivatives in $\boldsymbol{a}$. Since the r.h.s. of the first equation in (1) does not depend on time, its l.h.s. is obviously a Lagrangian invariant, whose scalar components are called the "Cauchy invariants". These invariants were much later interpreted as a consequence of Noether's theorem applied to a continuous symmetry of the Euler equation, called the relabelling invariance (see [20, Section 5.2]).

It was shown in [21,44] that the Cauchy invariants equations (1) imply analyticity of fluid trajectories for initial conditions that have certain rather weak regularity. The proof of analyticity is derived from an explicit recurrence relation for coefficients of the time-Taylor series for the Lagrangian displacement $\boldsymbol{\xi} = \boldsymbol{x} - \boldsymbol{a}$. It can also be used to construct a numerical Lagrangian method of very high order in time in both two dimensions (2D) and three dimensions (3D). Its time steps are only constrained by the radius of convergence of the Taylor series (typically, the inverse of the largest initial velocity gradient). For more details, see Section 2 and [21,44].

The paper is organised as indicated hereafter. In Section 2 we recall some of the known results about Cauchy's Lagrangian formulation and its application to the time-analyticity of the Lagrangian map. In Section 3 we describe the Cauchy–Lagrangian (CL) algorithm in detail: we begin with an overview and show that CL may be considered as a semi-Lagrangian algorithm of arbitrary high order (Section 3.1), present the interpolation technique for reverting to Eulerian coordinates at the end of each time step (Section 3.4) and show how to optimise the choices of the time step and of the order of the Taylor expansion truncation to minimise computational complexity (Section 3.5). Section 4 is devoted to testing the CL algorithm in the 2D case against various numerical methods, and to CPU benchmarks. Section 5 is a comparison of high-order time-Taylor expansions in Eulerian and Lagrangian coordinates: the Lagrangian method suffers much less from the rounding errors. In Section 6 we determine how quickly we have to decrease the time step in the CL method because the radius of convergence $R(t)$ of the time-Taylor series around time t generally shrinks as t increases, as more and more small-scale eddies (high spatial Fourier harmonics) are generated. Finally, in Section 7, we present concluding remarks and point out that the Cauchy–Lagrangian method is well adapted for certain other problems concerning ideal fluid flow.

2. Mathematical background

Here we recall some of the background material which is used to develop the Cauchy–Lagrangian method: derivation of the Cauchy invariants equations and the recurrence relation for time-Taylor coefficients from which follows the analyticity of the time-Taylor series (Section 2.1). We then present, in a heuristic way, a result allowing the determination of the radius of convergence of such Taylor series, an important new tool for analysing the output of Cauchy–Lagrangian computations (Section 2.2).

2.1. Cauchy's Lagrangian formalism for ideal incompressible fluid flow

The flow of an ideal incompressible fluid, described in Eulerian coordinates, is governed by the Euler equation

$$\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla) \boldsymbol{v} = -\nabla p, \qquad \nabla \cdot \boldsymbol{v} = 0, \tag{2}$$

where $\boldsymbol{v}(\boldsymbol{x}, t)$ is the velocity and $p(\boldsymbol{x}, t)$ the pressure (divided by the density, which in the incompressible case is constant). Here the spatial differentiation ∇ is performed in the Eulerian coordinates $\boldsymbol{x}$.

The Eulerian equation (2) is notorious for difficulties in its investigation, both analytically and numerically. It turns out that its Lagrangian analogue can be once integrated in time, which yields the Cauchy invariants equations (1). Since these equations are central to our numerical method, we briefly recall how they are derived.