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Finite-volume scheme for anisotropic diffusion

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ABSTRACT

In this paper, we apply a special finite-volume scheme, limited to smooth temperature distributions and Cartesian grids, to test the importance of connectivity of the finite volumes. The area of application is nuclear fusion plasma with field line aligned temperature gradients and extreme anisotropy. We apply the scheme to the anisotropic heat-conduction equation, and compare its results with those of existing finite-volume schemes for anisotropic diffusion. Also, we introduce a general model adaptation of the steady diffusion equation for extremely anisotropic diffusion problems with closed field lines.

1. Introduction

Most of the techniques to handle diffusion in anisotropic media are based on finite-volume or finite-element methods and revolve around handling the interpolation of the flux over the cell faces. A lot of work has been done on finite-volume schemes for the solution of diffusion problems on unstructured grids with discontinuous and anisotropic diffusion tensors. Here an important assumption in constructing the formulation of the cell-face fluxes is the continuity of the heat flux over the cell-faces, see for instance Edwards and Rogers [13], Breil and Maire [31] and Jacq et al. [25]. Vertex values are used in several cell-centered schemes to approximate the flux over the cell face, see e.g. Le Potier [28], Lipnikov et al. [30], Coudière et al. [10]. The vertex values are approximated with for instance continuity and monotonicity in mind. The vertex values may be defined explicitly but this requires some sort of dual grid, see e.g. Hermeline [22], Le Potier and Ong [29], Morel et al. [32]. Shashkov and Steinberg [34] put the flux values in the vertices and then average to the centers of the cell-faces. For a more detailed overview of finite-volume methods the reader is referred to the review paper by Droniou [11].

One of the most popular finite-volume methods for diffusion problems is the Multi-Point Flux Approximation (MPFA), a cell-centered finite-volume method commonly used for approximating diffusion with discontinuous tensors on distorted meshes, see e.g. Aavatsmark et al. [1,3–6] and Edwards and Rogers [13]. The method is robust in terms of diffusion tensor discontinuity. However, the resulting diffusion operator is often non-symmetric and formal accuracy cannot be maintained for higher levels of anisotropy. Aavatsmark [2] present a symmetric MPFA method and give formal proof of convergence. Friis et al. [20] and Edwards and Pal [12] present symmetric MPFA schemes for unstructured triangular and quadrilateral volumes respectively. The MPFA-method is also applied to multi-phase, multi-scale diffusion problems with grid refinement by Jenny et al. [26,27] and Hesse et al. [23].

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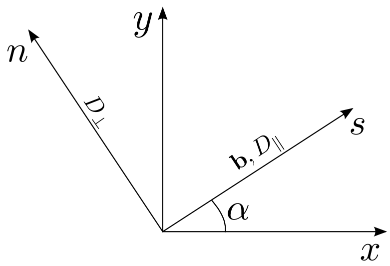

Fig. 1. Explanation of symbols.

Edwards and Zheng [14,16], Chen et al. [9] and Friis and Edwards [19] use a flux-continuous locally conservative MPFA-scheme called CVD-MPFA. The CVD-MPFA scheme includes the full diffusion tensor and extra auxiliary vertex unknowns and interface pressure value that are eliminated a priori using an auxiliary divergence condition and flux continuity conditions. They use M-matrix conditions and temperature continuity and flux continuity conditions to obtain a suitable quadrature point for the flux formulation. In case an M-matrix cannot be constructed, a Quasi-M-matrix is constructed which fulfills the former conditions as much as possible. The method does not guarantee M-matrices for high values of anisotropy, the latter is improved in the double parameter family of this scheme by Edwards and Zheng [15] and more generally the multi-parameter family by Edwards and Zheng [17]. These methods are robust for general tensor fields and unstructured grids.

What motivated Morel et al. [32], Breil and Maire [7], Hyman et al. [24], Edwards and Zheng [14,16] and others in developing flux(-normal) continuous schemes was grid robustness of finite-volume methods and finite-element methods in case of diffusion-tensor discontinuities. Van Es et al. [37] looked at the importance of alignment for a finite-difference method. In that paper several schemes are compared. The importance of internodal/volume continuity was expected because the formal accuracy for all schemes using series expansions was second order and a decisive effect of lower continuity at the boundaries was not visible in a local error analysis although it clearly mattered in terms of boundary treatment.

In this paper we propose and apply a finite-volume scheme that can change the connectivity between the volumes by changing the length of the cell faces with a free parameter. We apply both cell-face and vertex-centered flux points.

As before we approximate the anisotropic thermal diffusion, described by

$$\mathbf{q} = -\mathbf{D} \cdot \nabla T, \quad \frac{\partial T}{\partial t} = -\nabla \cdot \mathbf{q} + f, \tag{1}$$

where T represents the temperature, $\mathbf{b}$ the unit direction vector of the field line, f some source term and $\mathbf{D}$ the diffusion tensor. For a two-dimensional problem the diffusion tensor is given by

$$\text{unit direction vector:} \quad \mathbf{b} = (\cos\alpha, \sin\alpha)^T,$$
$$\mathbf{D} = (D_\parallel - D_\perp)\mathbf{b}\mathbf{b}^T + D_\perp \mathcal{I},$$
$$\mathbf{D} = \begin{pmatrix} D_\parallel b_1^2 + D_\perp b_2^2 & (D_\parallel - D_\perp) b_1 b_2 \\ (D_\parallel - D_\perp) b_1 b_2 & D_\perp b_1^2 + D_\parallel b_2^2 \end{pmatrix},$$

where $D_\parallel$ and $D_\perp$ represent the parallel and the perpendicular diffusion coefficient respectively.

We define x, y as the non-aligned coordinate system and s, n as the aligned coordinate system, see Fig. 1. We define the anisotropy as

$$\varsigma = \frac{D_\parallel}{D_\perp}.$$

In nuclear fusion plasma the level of anisotropy ς can be as large as 10^9, i.e. the diffusivity along the field lines is in the order of 10^9 times larger than the diffusivity perpendicular to the field lines. As the directions of the magnetic field lines vary continuously we have a full diffusion tensor throughout the domain. Variability of the diffusion coefficients is not considered in the current work.

2. Finite-volume schemes

All the schemes to be discussed formally have local second-order accuracy, which can be shown by carefully expanding the approximations using Taylor series, see Appendix A. However, as can be seen in the results of the test cases discussed in section 4, the accuracy of these methods may drop below their formal accuracy even though the test cases have $\mathcal{C}_\infty$ solutions and source functions. One important aspect that may be overlooked by the local analysis is the continuity between elements, or some equivalent property for finite differences. The symmetric scheme by Günter et al. [21] shows anisotropy independent results in case the diffusion tensor components are captured exactly by the staggered grid points.