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A Monte Carlo method for solving the one-dimensional telegraph equations with boundary conditions

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ABSTRACT

A Monte Carlo algorithm is derived to solve the one-dimensional telegraph equations in a bounded domain subject to resistive and non-resistive boundary conditions. The proposed numerical scheme is more efficient than the classical Kac's theory because it does not require the discretization of time. The algorithm has been validated by comparing the results obtained with theory and the Finite-difference time domain (FDTD) method for a typical two-wire transmission line terminated at both ends with general boundary conditions. We have also tested transmission line heterogeneities to account for wave propagation in multiple media. The algorithm is inherently parallel, since it is based on Monte Carlo simulations, and does not suffer from the numerical dispersion and dissipation issues that arise in finite difference-based numerical schemes on a lossy medium. This allowed us to develop an efficient numerical method, capable of outperforming the classical FDTD method for large scale problems and high frequency signals.

1. Introduction

Probabilistic methods based on Monte Carlo simulations have been used already to solve problems in Science and Engineering modeled by partial differential equations. The most important difference compared with the classical methods used so far rests on the possibility of computing the solution at a single point, being therefore essentially a meshless-type method. From the computational point of view this feature can be exploited advantageously over the classical methods. In fact, since no computational mesh is required, the well known memory constraints for solving large scale and high dimensional problems are minimized. Moreover, since the solution is computed by taking the average of independent calculations, the underlying algorithms are well suited for parallel computing [1]. However, unless one is interested to compute the solution at single points, as happens in some specific applications on the analysis of systems and networks, they are typically not competitive enough compared with classical numerical algorithms, when used to compute the solution at every point inside a given computational domain. This is basically due to the well-known slow convergence rate of the Monte Carlo method. An alternative consists in combining the Monte Carlo method with other classical techniques, such as the domain decomposition method, computing merely the solution in a few points along some chosen interfaces inside the domain. This method is called probabilistic domain decomposition method (PDD), and was successfully used to solve a variety of problems modeled by elliptic, and linear and semilinear parabolic partial differential equations [2,3].

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A key ingredient to implement a PDD method is to find a probabilistic representation of the solution. For linear elliptic equations, and parabolic equations, the probabilistic representation consists on the celebrated Feynman–Kac formula. In the specific field of electromagnetism, several accelerating techniques have been proposed in the literature to solve probabilistically electrostatic problems, such as the floating random walk [6], and walking on spheres [24,25], capable to speed up the computation of the solution by means of a variable time step size. For time-dependent electromagnetic problems, in particular for the time harmonic solution of the wave equation described by the scalar Helmholtz equation, a probabilistic representation was proposed in [5]. This representation is based on a suitable transformation, which allows to transform the original problem into two set of equations, one of them amenable to be solved using the Feynman–Kac formula.

In the general case of arbitrary hyperbolic partial differential equations, which model transport phenomena in Science and Engineering, a general probabilistic representation does not exist. However, there are two important exceptions. The first one is the Vlasov–Poisson system of equations, which appears to be of unquestionable interest in Plasma Physics. For such system of equations it was proposed already in [22] a probabilistic representation in the Fourier space, and a corresponding numerical method [4]. The second exception consists on the one-dimensional telegraph equations. For these equations the pioneering work by Kac [16,17], and Golstein [12], showed that a probabilistic representation in terms of a Poisson random walk can be readily derived. Basically, this was due to the fact that the telegraph equations can be seen as a wave equation with dissipation, and hence amenable to be described as an expected value of a suitable functional of a given stochastic process. Originally, the method proposed by Kac was derived exclusively to deal with problems in unbounded domains, as well as, zero initial velocity. More recently, in [15], it has been conveniently generalized to tackle the problem of arbitrary initial velocity. By exploiting the link between the wave equation and the telegraph equations, through a suitable random time, some useful numerical schemes for computing multidimensional fields in dispersive media have been introduced as well.

It is missing however a general probabilistic method capable of dealing with bounded domains, and moreover, from a computational point of view, of being competitive to be used as an efficient alternative to the widespread FDTD numerical method for electromagnetics. The goal of this paper is to fill these two gaps by proposing a novel Monte Carlo method to solve the telegraph equations governing the evolution of voltage and current in a two-wire transmission line. We show in this article how the method can be extended with suitable boundary conditions, such as, feeding the line with a non-ideal voltage source with internal resistance at one end, and terminating it with a non-resistive load at other.

It is worth to remark that the telegraph equations are formally the same type of equations as the Maxwell equations in 1D, therefore the method proposed in this paper can be trivially extended to deal with such a system of equations.

Here it is the outline of the paper. The probabilistic algorithm for solving the telegraph equations is presented in Section 2 for the unbounded case. In particular, we discuss the computational advantages offered by such an algorithm compared with the classical one. In Section 3 the method is extended to cope with bounded domains subject to suitable boundary conditions. Section 4 is devoted to numerical examples, where the results obtained with our new algorithm are compared with the results obtained with the classical FDTD method and theoretical solutions based on Fourier analysis. Finally, to close the paper, in Section 5 we summarize the more relevant findings and possible extensions of this work.

2. Probabilistic formulation of telegraph equations

The system of first-order partial differential equations governing the evolution of the voltage $u(x,t)$ and current $i(x,t)$ in a general two-wire transmission line (also known as telegraphist's equations) are given by

$$\begin{aligned}\frac{\partial u}{\partial x} &= -L\frac{\partial i}{\partial t} - R\,i,\\ \frac{\partial i}{\partial x} &= -C\frac{\partial u}{\partial t} - G\,u,\end{aligned} \tag{1}$$

where R, L, G, and C, are the resistance, inductance, conductance, and capacitance per unit length of the line, respectively. For simplicity, let us assume that no boundary conditions are imposed and that we choose arbitrary initial conditions $u(x,0) = u_0(x)$, and $i(x,0) = i_0(x)$. Recall that the Maxwell equations for the electric and magnetic fields $\mathbf{E}$, and $\mathbf{H}$ in a lossy media, assuming a x-directed z-polarized TEM wave ($H_x = E_x = 0$), with no variations in the y and z direction, are given by the one-dimensional equations

$$\begin{aligned}\frac{\partial E_y}{\partial x} &= -\mu\frac{\partial H_z}{\partial t},\\ \frac{\partial H_z}{\partial x} &= -\varepsilon\frac{\partial E_y}{\partial t} - \sigma\,E_y,\end{aligned} \tag{2}$$

where ε, and μ are the electric permittivity and magnetic permeability, respectively, and σ the electric conductivity. Note that both equations, the telegraph equations and the Maxwell equations, are formally the same type of equations, obtaining one from the other by simply replacing the voltage u by the electric field E_y, the current i by the magnetic field H_z, and the parameters L by μ, C by ε, G by σ, and finally setting $R = 0$. Through the transformation $f(x,t) = (u + R_0\,i)/2$, and $b(x,t) = (u - R_0\,i)/2$, where $R_0 = \sqrt{L/C}$ is the constant characteristic resistance, Eqs. (1) can be rewritten as