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The panel-clustering method for the wave equation in two spatial dimensions ☆

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ABSTRACT

We consider the numerical solution of the wave equation in a two-dimensional domain and start from a boundary integral formulation for its discretization. We employ the convolution quadrature (CQ) for the temporal and a Galerkin boundary element method (BEM) for the spatial discretization. Our main focus is the sparse approximation of the arising sequence of boundary integral operators by panel clustering. This requires the definition of an appropriate admissibility condition such that the arising kernel functions can be efficiently approximated on admissible blocks. The resulting method has a complexity of $\mathcal{O}\left(N\left(N+M\right)q^{4+s}\right)$, $s\in\{0,1\}$, where N is the number of time points, M denotes the dimension of the boundary element space, and $q=\mathcal{O}\left(\log(NM)\right)$ is the order of the panel-clustering expansion. Numerical experiments will illustrate the efficiency and accuracy of the proposed CQ-BEM method with panel clustering.

1. Introduction

The efficient and reliable simulation of scattered waves in unbounded exterior domains is a numerical challenge and the development of *fast* numerical methods is far from being matured. We are here interested in a boundary integral formulation of the problem to avoid the use of an artificial boundary with approximate transmission conditions [27,2,11,18,8] but allow to recast the problem (under certain assumptions which will be detailed later) as an integral equation on the surface of the scatterer. As our model problem we consider the homogeneous wave equation

$$\begin{aligned} &\partial_t^2 u = \Delta u && \text{in } \Omega\times(0,T)\,,\\ &u(\cdot,0)=\partial_t u(\cdot,0)=0 && \text{in } \Omega\,,\\ &u=g && \text{on } \Gamma\times(0,T)\,, \end{aligned} \tag{1}$$

where $\Omega\subset\mathbb{R}^2$ is either a bounded domain or the exterior of a bounded domain and $\Gamma:=\partial\Omega$. The methods for solving this problem can be split into a) *frequency domain* methods where an incident plane wave at prescribed frequency excites a scattered field and a time periodic ansatz reduces the problem to a purely spatial Helmholtz equation and b) *time-domain* methods where the excitation is allowed to have a broad temporal band width and, possibly, an a-periodic behavior with respect to time.

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In our paper we will focus on time-domain methods for the wave equation which is particularly important to model electric or acoustic systems shortly after they are "switched on", i.e., before the system has reached a time-harmonic steady state.

The formulation of (1) as a space–time integral equation by the *retarded acoustic single layer potential* can be written in the form

$$\int_0^t \int_\Gamma k\left(\|x-y\|, t-\tau\right) \varphi\left(y,\tau\right) d\Gamma_y d\tau = g\left(x,t\right) \qquad \forall (x,t) \in \Gamma \times (0,T), \tag{2}$$

where k is the fundamental solution for the acoustic wave equation.

Among the most popular methods for discretizing this equation are: a) the *convolution quadrature* (CQ) method [33,34,24,32,6,13] and b) the direct *space–time Galerkin discretization* of (2) (see, e.g., [5,19,20,40,41,45]).

The goal of this paper is to present fast solution methods for solving the wave equation in two spatial dimensions via (2) and to base the discretization on the CQ-method. The kernel function is given by applying the inverse Laplace transform $\mathcal{L}^{-1}$ to the *transfer function* K:

$$k\left(r,\bullet\right) := \mathcal{L}^{-1}\left(K\left(r,\bullet\right)\right) = \frac{1}{2\pi\,\mathrm{i}} \int_{I_\sigma} \mathrm{e}^{z\bullet} K\left(r,z\right) dz \quad \text{with} \quad K\left(r,z\right) := \frac{1}{2\pi} K_0\left(rz\right)$$

along a vertical contour

$$I_\sigma = \sigma + \mathrm{i}\mathbb{R} \quad \text{for some } \sigma > 0, \tag{3}$$

and K_0 being the modified Bessel function (see, e.g., [1, Sec. 9.6]). For this problem, we will introduce the panel-clustering method for the sparse representation of the discrete CQ-BEM operators. For problems in *three* spatial dimensional domains $\Omega \subset \mathbb{R}^3$ and Γ being a two-dimensional Lipschitz manifold, a fast version of the convolution quadrature *with BDF2* for the temporal discretization has been developed in [25,29,7]. Although there is a reduction with respect to memory and CPU time compared to the conventional approach the arising method is not of optimal complexity $\mathcal{O}\left(NM\right)$ (modulo additional factors depending only logarithmically on N and M), where N denotes the number of time steps and M is the dimension of the boundary element space. In this paper, we consider the panel-clustering method for the CQ-BEM *with BDF1* in two spatial dimensions and prove the log-linear scaling with respect to the total number of unknowns for both, CPU time and memory requirement.

It is well known that the fundamental solution of a second order partial differential equation (PDE) in *even* (spatial) dimensions is more complicated than in *odd* dimensions and new techniques for its approximation have to be developed. The speedup and memory savings of the resulting method is substantial and more significant than for the methods described in [25,29]: more precisely, the storage and computational complexity is $\mathcal{O}\left(N\left(N+M\right)q^{4+s}\right)$ with $q = \mathcal{O}\left(\log\left(NM\right)\right)$ and $s \in \{0,1\}$ instead of $\mathcal{O}\left(NM^2\right)$ for the classical CQ-BEM method. If we assume $M \sim N$, we obtain an optimal complexity (up to logarithmic terms) with respect to the total number of freedoms. We note in passing that boundary integral equations can be used to define transparent transmission conditions at artificial boundaries for wave propagation problems; the above mentioned CQ-BEM method has been proposed in [12] for an efficient discretization of such conditions. The new method we propose here also allows for a sparse realization of such exact non-local transmission conditions, where the complexity grows log-linearly with respect to the total number of unknowns $N_{\mathrm{tot}} := NM$.

Our new panel-clustering method for the two-dimensional wave equation requires the generalization and combination of quite different discretization techniques such as convolution quadrature, boundary element method, and panel clustering for complicated kernel functions. We recall the definitions of the basic algorithms in order to keep the presentation self contained and to estimate the complexity of the different steps of the algorithm. The paper is organized as follows.

In Section 2, we formulate the convolution quadrature method for the two-dimensional wave equation and introduce the boundary element method for its spatial discretization.

In Section 3, the panel-clustering method based on an abstract *admissibility condition* is introduced, while Section 4 is devoted to its implementation. This algorithmic formulation of the method will also play an essential role for the complexity estimates of the method.

The error analysis is carried out in Section 5. We employ functional-type estimates for certain derivatives of modified Bessel and exponential functions, recently presented by the authors in [15], to derive a non-standard admissibility condition for the panel-clustering approximation of the arising kernel functions. The local approximation error will be estimated and used for the stability and consistency analysis.

In Section 6, we will prove that the storage and computational complexity of the resulting CQ-BEM method with panel clustering is $\mathcal{O}\left(N\left(N+M\right)q^{4+s}\right)$, where $q = \mathcal{O}\left(\log\left(NM\right)\right)$ and $s \in \{0,1\}$.

We will present the results of numerical experiments in Section 7 which demonstrate that the theoretical complexity and error estimates are sharp for the considered model problems.