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Mass-corrections for the conservative coupling of flow and transport on collocated meshes

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ABSTRACT

Buoyancy-driven flow models demand a careful treatment of the mass-balance equation to avoid spurious source and sink terms in the non-linear coupling between flow and transport. In the context of finite-elements, it is therefore commonly proposed to employ sufficiently rich pressure spaces, containing piecewise constant shape functions to obtain local or even strong mass-conservation. In three-dimensional computations, this usually requires nonconforming approaches, special meshes or higher order velocities, which make these schemes prohibitively expensive for some applications and complicate the implementation into legacy code. In this paper, we therefore propose a lean and conservatively coupled scheme based on standard stabilized linear equal-order finite elements for the Stokes part and vertex-centered finite volumes for the energy equation. We show that in a weak mass-balance it is possible to recover exact conservation properties by a local flux-correction which can be computed efficiently on the control volume boundaries of the transport mesh. We discuss implementation aspects and demonstrate the effectiveness of the flux-correction by different two- and three-dimensional examples which are motivated by geophysical applications.

1. Introduction

In the last decade, the importance of using locally conservative methods for coupled flow and transport problems was emphasized by many researchers in the field of computational fluid dynamics; cf., e.g., the discussions in [1,2]. Especially in situations where we consider a long-term nonlinear and bidirectional coupling between incompressible flow and transport, the use of compatible (i.e., locally conservative) algorithms is strongly recommended, to avoid a severe violation of physical conservation principles due to the amplification of spurious sources and sinks in the flow field. Such compatible algorithms often treat the subproblems separately, by ensuring that the discrete schemes yield local or even strongly conservative velocities, which are used to compute convective fluxes in a locally conservative transport scheme. Suitable schemes of arbitrary order for the transport equation are readily available [3,4]. However, the question of conservation for the flow part is a delicate one: While it is straightforward to show local or even strong conservation properties for mixed finite element schemes with piecewise discontinuous pressure spaces, it is difficult to find schemes which are at the same time LBB stable [5]. Hence, it is a long-standing belief that improved (local) conservation properties require either penalization techniques

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[6,2] or are limited to non-conforming, stabilized or higher order mixed finite element schemes involving velocity spaces which are rich enough to retain stability when paired with piecewise discontinuous pressures [7–9].

In this work, we follow a different strategy which does not rely on flow discretizations that directly yield locally conservative velocities. We rather build upon lean stabilized discretizations for which a mature machinery of fast solvers exists, and we employ the framework established in [10] to recover conservative fluxes for the transport. Hence, in contrast to most related work, we do not treat the coupled problem as two distinct subproblems which have to satisfy local conservation requirements, but we rather directly consider the conservative coupling between both schemes in their derivation to develop a novel concept suitable for implementation into finite element solvers based on piecewise linear shape functions.

We consider a class of applications which is concerned with a coupling between incompressible flow and transport equations. Our motivation is the study of large-scale geophysical phenomena such as mantle convection physics, which occur on a wide range of time and length scales, requiring long term simulations of large-scale coupled systems; cf. e.g. [11–13].

As a model problem, we choose a generalized Stokes-type problem, serving as a simplified variant of the more complex physics used in the simulation of convection in the Earth's mantle [12]. For simplicity, we use the Boussinesq approximation with constant conductivity, and we neglect inertia and internal heating. This results in the following set of non-dimensional balance equations, governing the conservation of momentum, mass and energy in a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$:

$$-\operatorname{div}(2\mu(\vartheta, z)\,\operatorname{sym}\nabla\mathbf{u}) + \nabla p = -\vartheta\,\widehat{\mathbf{e}}, \qquad \text{in } \Omega \times (0, t_{\text{end}}],\ t_{\text{end}} > 0, \tag{1a}$$

$$\operatorname{div}\mathbf{u} = 0, \qquad \text{in } \Omega \times (0, t_{\text{end}}], \tag{1b}$$

$$\partial_t\vartheta + \operatorname{div}(\vartheta\mathbf{u} - \text{Ra}^{-1}\nabla\vartheta) = 0. \qquad \text{in } \Omega \times (0, t_{\text{end}}]. \tag{1c}$$

Here $\mathbf{u}$ represents the velocity field, p the pressure, ϑ the temperature, and $\widehat{\mathbf{e}}(\mathbf{x})$ is the unit-length vector aligned with gravitational forces. The normalized viscosity μ potentially depends on the temperature ϑ and/or the depth $z(\mathbf{x})$ which is a domain-specific function of $\mathbf{x} := (x_1, \ldots, x_d) \in \Omega$, and the Rayleigh-number $\text{Ra} > 0$ determines the presence and strength of convection caused by the gravitational forcing term which is given by $-\vartheta\widehat{\mathbf{e}}$. In mantle convection models, we usually deal with situations where $\text{Ra} \gg 1$ (a typical value for numerical models is $4.2 \cdot 10^7$), i.e., the energy transport is strongly dominated by convection [12].

To make the problem well-posed, we consider an initial temperature field $\vartheta(t = 0) = \vartheta^0$ and homogeneous Neumann boundary conditions $\text{Ra}^{-1}\nabla\vartheta \cdot \mathbf{n} = 0$ on the whole boundary $\partial\Omega$ of the computational domain, where $\mathbf{n}$ denotes an outward-pointing unit-normal vector. For the Stokes part, we set no-slip Dirichlet-boundary conditions $\mathbf{u} = \mathbf{0}$, which obviously satisfy the compatibility condition $\int_{\partial\Omega} \mathbf{u} \cdot \mathbf{n}\, ds = 0$. The extension of the framework to more general boundary conditions is fairly straightforward and not considered here for the sake of brevity. We shall however present results obtained with other types of boundary conditions when we discuss the numerical examples in Section 3.

Similarly structured models have applications in other fields where mass conservation is important, e.g., in ice sheet modeling [14–16], or in two-phase flow, where, instead of a temperature field, the evolution of an interface is considered in (1c); cf. e.g. [17–20]. Although we consider only those physical effects which are important for the following presentation, we would like to emphasize that the extension to more realistic situations, involving for instance inertial terms and additional transport phenomena, is possible. Using a suitable reformulation of the model equations in terms of the conservative momentum density $\rho\mathbf{u}$ instead of the velocity $\mathbf{u}$, we can also incorporate slight compressibility effects. This allows the generalization of the approach proposed in this paper towards more sophisticated convection models such as the anelastic liquid approximation of mantle convection [12]. Similar ideas were previously proposed in the context of compressible flow [21].

The outline of this work is as follows: In the following Section 2, we shall introduce the necessary notation and the variational form of our conservative coupling strategy. We furthermore briefly discuss the extension to other stabilizations for the Stokes part of the problem and higher order time-integration schemes for the energy equation. Then in Section 3, we show a series of numerical results which are motivated by standard geophysical benchmarks. We demonstrate the performance of the novel coupled scheme in several examples by comparing it to a non-conservatively coupled scheme. Finally, in Appendix A, we discuss some details concerning the implementation of the proposed approach in three-dimensional settings.

2. Conservative coupling

In the following, we shall restrict ourselves to first-order explicit time-stepping for simplicity; extensions to higher order time-integration shall be discussed in Section 2.3. By approximating $\partial_t\vartheta \approx (\vartheta^{n+1} - \vartheta^n)/\Delta t^n$, we obtain a desirable decoupling of the problem into an instantaneous constraint given in form of a generalized Stokes equation

$$-\operatorname{div}(2\mu(\vartheta^n, z)\,\operatorname{sym}(\nabla\mathbf{u}^n)) + \nabla p^n = -\vartheta^n\,\widehat{\mathbf{e}}, \qquad \text{in } \Omega, \tag{2a}$$

$$-\operatorname{div}\mathbf{u}^n = 0, \qquad \text{in } \Omega, \tag{2b}$$

and a semi-discretization of an equation of transport-type

$$\vartheta^{n+1} = \vartheta^n - \Delta t^n \operatorname{div}(\vartheta^n\,\mathbf{u}^n - \text{Ra}^{-1}\nabla\vartheta^n) \qquad \text{in } \Omega, \tag{2c}$$