Journal of Computational Physics 305 (2016) 403–422

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

A fully spectral methodology for magnetohydrodynamic calculations in a whole sphere

P. Marti [a,b,*], A. Jackson [b]

[a] Department of Applied Mathematics, University of Colorado, Boulder, CO 80309, USA
[b] Institute of Geophysics, ETH Zurich, Zurich 8092, Switzerland

ARTICLE INFO

Article history:
Received 19 May 2015
Received in revised form 28 October 2015
Accepted 31 October 2015
Available online 4 November 2015

Keywords:
Sphere
CFL condition
HPC
Spectral method
Dynamo
Orthogonal poylnomials
MHD

ABSTRACT

We present a fully spectral methodology for magnetohydrodynamic (MHD) calculations in a whole sphere. The use of Jones–Worland polynomials for the radial expansion guarantees that the physical variables remain infinitely differentiable throughout the spherical volume. Furthermore, we present a mathematically motivated and systematic strategy to relax the very stringent time step constraint that is present close to the origin when a spherical harmonic expansion is used for the angular direction. The new constraint allows for significant savings even on relatively simple solutions as demonstrated on the so-called full sphere benchmark, a specific problem with a very accurately-known solution. The numerical implementation uses a 2D data decomposition which allows it to scale to thousands of cores on present-day high performance computing systems. In addition to validation results, we also present three new whole sphere dynamo solutions that present a relatively simple structure.

1. Introduction

A number of the celestial bodies in our solar system are believed to have today, or at least had at an earlier stage, a fluid core generating their magnetic field; an overview can be found in [31]. In this paper we consider the canonical problem of fluid motion and dynamo action (spontaneous magnetic field generation) in a whole sphere. Although modeling of planetary cores in a spherical shell geometry (i.e. with an impenetrable inner boundary) is relatively mature, the whole sphere geometry has received considerably less attention. It is possible that Venus and Mars do not possess a solid inner core, and certainly at an early stage the Earth did not possess an inner core. Paleomagnetic measurements provide evidence that the magnetic field of the Earth existed at least 3.5 billion years ago (see e.g. [33]), so the majority of its life has been spent without an inner core.

There has been great progress in understanding and characterizing planetary dynamos in the last decades. Advances have been made both from a theoretical point of view (see e.g. [26]) and by tackling the problem by a numerical approach, starting from the first Boussinesq simulations [11] to more recent results requiring massive computing power (e.g. [32]). Most of the current studies are performed for a spherical shell geometry, recognizing the presence of an inner core in the present-day Earth. The presence of an inner core plays an important role as it provides a strong driving force through the latent heat that is released during its crystallization. In the absence of an inner core, this driving force is not available and

* Corresponding author at: Department of Applied Mathematics, Engineering Center, Room ECOT 247, University of Colorado, Boulder, CO 80309-0515, USA.

E-mail addresses: philippe.marti@colorado.edu (P. Marti), ajackson@ethz.ch (A. Jackson).

http://dx.doi.org/10.1016/j.jcp.2015.10.056
0021-9991/

dynamo action through thermal convection becomes more difficult. Assuming the mechanical forcings, for example due to precession, are not important, a thermally driven full sphere dynamo relies on internal heating by radiogenic sources and secular cooling.

From a theoretical and computational point of view, the full sphere geometry has an additional advantage. The absence of an inner core simplifies the geometry and removes one control parameter, the ratio of inner core to outer core radii. Simulations do not need to resolve the sharp Stewartson layers tangent to the inner core due to a small differential rotation between the outer and inner boundary (see e.g. [9]), thus reducing the computational requirements. We believe that the study of the whole sphere dynamo problem can bring great insight into the underlying physics by "filtering out" part of the underlying complexity.

We solve the nondimensional magnetohydrodynamics equations in the Boussinesq approximation in a full sphere geometry. To cast the dimensional equations into a nondimensional form we used: the core radius r_o as the characteristic length scale d, the time is scaled by the magnetic diffusion time $\frac{d^2}{\eta}$ while the magnetic field is rescaled by $\sqrt{2\Omega\rho\mu_0\eta}$. Here ρ is the density, Ω the rotation rate, μ_0 is the permeability of free space and η is the magnetic diffusivity. Using these dimensional scales leads to the following set of nondimensional MHD equations:

$$\left(E_m\,\partial_t - E\,\nabla^2\right)\boldsymbol{u} = -\nabla\hat{p} + E_m\,\boldsymbol{u}\wedge(\nabla\wedge\boldsymbol{u}) + (\nabla\wedge\boldsymbol{B})\wedge\boldsymbol{B} + q\,Ra\,C\,\boldsymbol{r} - \hat{\boldsymbol{z}}\wedge\boldsymbol{u} \tag{1}$$

$$\left(\partial_t - \nabla^2\right)\boldsymbol{B} = \nabla\wedge(\boldsymbol{u}\wedge\boldsymbol{B}) \tag{2}$$

$$\left(\partial_t - q\,\nabla^2\right)C = S - \boldsymbol{u}\cdot\nabla C \tag{3}$$

$$\nabla\cdot\boldsymbol{u} = 0 \tag{4}$$

$$\nabla\cdot\boldsymbol{B} = 0 \tag{5}$$

where $\boldsymbol{u}$ is the non-dimensional velocity, $\boldsymbol{B}$ the non-dimensional magnetic field, $\hat{p}$ is pressure and C the non-dimensional codensity (see e.g. [5]). The S term in the transport equation (3) represents the heat sources, and $\hat{\boldsymbol{z}}$ is the direction along the rotation axis.

There are four non-dimensional control parameters: the Ekman number E which is the ratio between the viscous forces and the Coriolis force, the Roberts number q which is the ratio between thermal diffusivity and magnetic diffusivity, a modified Rayleigh number Ra which characterizes the strength of buoyancy and the magnetic Ekman number E_m which is the ratio between magnetic diffusion and the Coriolis force, all defined as follows:

$$E = \frac{\nu}{2\Omega d^2}; \qquad q = \frac{\kappa}{\eta}; \qquad Ra = \frac{\alpha\beta g_0 d}{2\Omega\kappa}; \qquad E_m = \frac{\eta}{2\Omega d^2} \tag{6}$$

In these definitions ν is the viscosity, κ the thermal diffusivity, α the thermal expansivity, g_0 the gravity at the core radius (we assume a linearly increasing gravity with r) and $\beta = S/(3\kappa)$.

While our present focus is on the thermally driven dynamo problem, we would like to stress that the same approach can and has been implemented for mechanically forced flows, e.g. precession and libration driven flows in spherical geometry (see e.g. [15]). The thermally driven problem does however, exhibit all the numerical issues that have had to be addressed.

The layout of the paper is as follows: in §2 we present the numerical model that is used to solve the Boussinesq dynamo equations including the time integration and an approach to avoid the stringent time step constraints when one gets close to the origin. In §3 we describe the efficient parallelization approach we implemented to allow our simulation to be run on a few thousands of CPUs. Finally in §4 we give validation results and discuss general properties of our simulations.

2. Numerical model

2.1. Discretization

The full sphere geometry is geometrically simpler than a spherical shell. For a numerical approach, it does nevertheless come with a complication. We naturally choose a spherical polar coordinate system (r, θ, ϕ) where r is radius, θ is colatitude and ϕ is longitude. As opposed to the spherical shell, the origin of the spherical coordinate system is present within the integration volume for the numerical solver. Due to the definition of the coordinate system, it is an artificial singularity which can lead to numerical instabilities if not treated carefully (see e.g. [4,21]), and much emphasis in this paper is placed on its proper treatment. As a result of our treatment, all our defined fields $(\boldsymbol{u}, \boldsymbol{B}, C)$ are infinitely differentiable everywhere and completely unaware of the presence of the coordinate system singularity. To achieve high accuracy and fast convergence, we implemented a fully spectral simulation method. The spherical geometry provides a natural choice for the expansion in the angular component, namely the spherical harmonics $\mathcal{Y}_l^m(\theta, \phi)$. Following the traditional route in spherical dynamo simulations, we use the Schmidt quasi-normalized complex spherical harmonics (see [28]) such that

$$\mathcal{Y}_l^m(\theta, \varphi) = P_l^m(\cos\theta)e^{im\varphi} = \sqrt{\frac{(l-m)!}{(l+m)!}}\,\widehat{P}_l^m(\cos\theta)e^{im\varphi}. \tag{7}$$