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On regularizations of the Dirac delta distribution ☆

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A B S T R A C T

In this article we consider regularizations of the Dirac delta distribution with applications to prototypical elliptic and hyperbolic partial differential equations (PDEs). We study the convergence of a sequence of distributions $\mathcal{S}_H$ to a singular term $\mathcal{S}$ as a parameter H (associated with the support size of $\mathcal{S}_H$) shrinks to zero. We characterize this convergence in both the weak-$*$ topology of distributions and a weighted Sobolev norm. These notions motivate a framework for constructing regularizations of the delta distribution that includes a large class of existing methods in the literature. This framework allows different regularizations to be compared. The convergence of solutions of PDEs with these regularized source terms is then studied in various topologies such as pointwise convergence on a deleted neighborhood and weighted Sobolev norms. We also examine the lack of symmetry in tensor product regularizations and effects of dissipative error in hyperbolic problems.

1. Introduction

Many phenomena in the physical sciences are modeled by partial differential equations (PDE) with singular source terms. The solutions of such PDE models are often studied using numerical approximations. In some computational approaches, the singular source terms are represented *exactly*, such as in [1,7,8,35]. A more common approach is to approximate the source term using some regularized function, and then obtain the numerical solution using a discretization of the PDE with the approximate source. One prominent example of the utility of singular sources in applications is the immersed boundary method [28], wherein a Dirac delta distribution supported on an immersed fiber or surface is used to capture the two-way interaction between a dynamically evolving elastic membrane and the incompressible fluid in which it is immersed. In immersed boundary simulations, the Dirac delta is replaced by a continuous approximation that is designed to satisfy a number of constraints that guarantee certain desirable properties of the analytical and numerical solution. Related approximations are also employed in connection with the level set method [27] and vortex methods [3,10].

Suppose we represent the original problem of interest in an abstract form as follows:

Problem 1: Find u such that

$$\mathcal{L}(u) = \mathcal{S}, \tag{1a}$$

where $\mathcal{L}$ is a PDE operator and $\mathcal{S}$ is a distribution which is used to model a singular source.

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Then let $\mathcal{S}_H$ denote some approximation of $\mathcal{S}$ and consider the associated problem

Problem 2: Find u_H such that

$$\mathcal{L}(u_H) = \mathcal{S}_H. \tag{1b}$$

Here, $H > 0$ is some small parameter for which $\mathcal{S}_H \to \mathcal{S}$ in some sense as $H \to 0$ (a sense that will be made concrete later on). One may then apply an appropriate numerical scheme (e.g., finite difference, finite volume, finite element, spectral, etc.) with a discretization parameter $h > 0$, thereby obtaining two discrete solution approximations: u_h to u in *Problem 1*; and $u_{H,h}$ to u_H in *Problem 2*. We are free, of course, to pick one numerical scheme for *Problem 1* and a different scheme for *Problem 2*. If the numerical schemes are suitably well-chosen, then both $u_h \to u$ and $u_{H,h} \to u_H$ as $h \to 0$.

In practical computations, it may not be possible to construct u_h. Indeed, it is typically only $u_{H,h}$ that is computed, by first prescribing some approximation to the source term and then discretizing the PDE with the approximate source. Ideally, what we hope to obtain is that as both $h, H \to 0$, the discrete approximant $u_{H,h} \approx u$. In the immersed boundary method, for example, the source term $\mathcal{S}$ is a line source and both the approximation of the source term and the discretization of the PDE are performed with reference to the same underlying spatial grid, so that parameters H and h are identical. Convergence of $u_{H,h} \to u$ in the context of the immersed boundary method has been the subject of detailed analysis in the works of Liu and Mori [23,24,26]. However, these authors only focus on convergence of the discrete regularizations and do not consider $u_H \to u$.

In this article we are concerned primarily with two questions:

Question 1. How do we construct 'good' approximations $\mathcal{S}_H$ to $\mathcal{S}$?

Question 2. How does the choice of approximation $\mathcal{S}_H$ affect the convergence of $u_{H,h} \to u$?

Before we can formulate answers to the above, we have to first answer the two related questions:

Question 3. What form of convergence should be used to examine $\mathcal{S}_H \to \mathcal{S}$?

Question 4. What form of convergence should be used to examine $u_{H,h} \to u$?

In this paper, we restrict our attention to the particular case of $\mathcal{S} = \delta$ which denotes the well-known point source distribution (or Dirac delta distribution) having support at the origin.

Questions 1 and *2* are fairly well-studied in some contexts [16,23,24,6,32,31,30,36] but *Questions 3* and *4* have not been the subject of much scrutiny in the literature. A common approach for approximating $\mathcal{S}_H$ (via regularization) is to construct a *discrete regularization* that is tailored to specific quadrature methods. Waldén [34] presents an analysis of discrete approximations of the delta distribution, restricting his attention to applications to PDEs in one dimension. Tornberg and Engquist [32] analyze discrete approximations to the delta distribution in multiple dimensions with compact support and draw a connection between the discrete moment conditions and the order of convergence of the solution of a PDE with the discrete $\mathcal{S}_H$ as source term. They also consider approximations of line sources using a singular source term or a collection of delta distributions in a chain. The analyses of Tornberg [30] and Tornberg and Engquist [31] for the discrete approximations $\mathcal{S}_H$ rely on the choice of mesh and quadrature rules. They also restrict $H = \mathcal{O}(h)$ and compare $u_{H,h}$ directly to u so that, $\mathcal{S}_H$ is based on the numerical method used to compute $u_{H,h}$. More recently, Suarez et al. [29] considered regularizations of the delta distribution that are tailored to spectral collocation methods for the solution of hyperbolic conservation laws. Their approach to constructing polynomial regularizations using the Chebyshev basis has a similar flavor to our approach, as will be seen in Section 3. In a different approach, Benvenuti et al. [5] study the case of regularizations that are not compactly supported but have rapidly decaying Fourier transforms in the context of extended finite element methods (XFEM) [4]. The authors demonstrate that such regularizations lead to lower numerical errors since they can be integrated using common quadrature methods such as Gauss quadrature.

In this article we demonstrate firstly how to develop regularizations $\mathcal{S}_H$ independent of the choice of numerical discretization. For example, in answering *Question 1* we derive *piecewise smooth* approximations $\mathcal{S}_H$. We can then examine the intermediate errors $\|u - u_H\|_X$ and $\|u_H - u_{H,h}\|_X$ and use the triangle inequality to give a bound on

$$\|u - u_{H,h}\|_X \leq \underbrace{\|u - u_H\|_X}_{\text{regularization error}} + \underbrace{\|u_H - u_{H,h}\|_X}_{\text{discretization error}}, \tag{2}$$

where $\|\cdot\|_X$ refers to a suitably chosen norm; the choice of norms is discussed below. For fixed $H > 0$ the discretization errors $\|u_{H,h} - u_H\|_X$ are analyzed using properties of the numerical scheme and regularity of solutions of *Problem 2*. The resulting discretization errors are well-understood for specific problems and specific schemes, and so we focus our attention here on the regularization errors.

We propose a unified approach for construction and analysis of regularizations. This has three advantages: First, we are able to provide a simple strategy for constructing new regularizations suitable for a given application (and not constrained to a specific numerical method for that application). Second, our framework is flexible and allows us to study the effect of