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Adaptive Haar wavelets for the angular discretisation of spectral wave models

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ABSTRACT

A new framework for applying anisotropic angular adaptivity in spectral wave modelling is presented. The angular dimension of the action balance equation is discretised with the use of Haar wavelets, hierarchical piecewise-constant basis functions with compact support, and an adaptive methodology for anisotropically adjusting the resolution of the angular mesh is proposed. This work allows a reduction of computational effort in spectral wave modelling, through a reduction in the degrees of freedom required for a given accuracy, with an automated procedure and minimal cost.

1. Introduction

A wide range of different numerical models are now available which can be used for the study of wave generation and propagation. These can be split into two main categories: phase-resolving and phase-averaging [1]. Phase-resolving models, such as potential flow, mild-slope, Boussinesq and full 3D Navier–Stokes models represent the sea surface elevation in space and time and accurately account for the non-linear processes. They are, however, computationally expensive and, thus, restricted to relatively small scale applications. Phase-averaging models are based on a spectral description of the waves and though the non-linearities are represented by parametrised formulations, they are cheap enough to be used on larger problem domains.

Spectral wave modelling first appeared after the introduction of the wave energy spectrum by Pierson [2] and the introduction of the energy balance equation by Gelci [3]. Based on linear wave theory, the sea surface elevation is composed of a superposition of harmonic wave components and the energy spectrum $E(x, y, f, \theta, t)$ represents the energy content over frequencies f and directions θ, in space (x, y) and time t. All of the important characteristics of the sea surface, such as the significant wave height or the mean period, can then be seen as statistical parameters of the spectrum and derived from various combinations of its moments $m_n = \int\int f^n E(f, \theta) df d\theta$ [4].

The energy spectrum is calculated based on the conservation of energy in an Eulerian framework. A kinematic part representing the propagation of wave energy is balanced with a set of source terms which represent wind generation, non-linear energy transfers and wave dissipation. Various spectral wave models have been developed from as early as the

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1960s. So-called "first-generation models" did not account (or loosely accounted) for non-linear wave energy interactions. "Second-generation" models used a simplified parametrised form for these interactions, restricting the shape of the spectrum. A thorough review of these early models can be found in [5].

The next milestone in spectral wave modelling came from the WAMDI group [6] with the introduction of WAM, a model with improved formulations for the source terms and no a-priori restriction on the shape of the spectrum. This framework was coined as "third-generation" wave modelling and followed by rapid developments. Currents were included in the formulation by rewriting the governing equation in terms of the action density $A = E/f$, which is conserved in a relative frame of reference moving with the current [7]. The action balance equation was then extended to account for shallow water propagation, such as shoaling and refraction and shallow water non-linear processes, such as triads and depth induced breaking. For a thorough review of the most notable developments the reader can refer to [8] and [9]. Today the most widely used third generation models by the community are WAM [6] and WAVEWATCHIII [10] for global scales and SWAN [11] for coastal applications.

The source parameterisations and the numerical schemes for spectral wave models are still an active field of research. The last five years, for example, has seen the error levels for the prediction of significant wave heights and mean periods in the middle of the ocean drop by 20% and 30% respectively [12]. A further reduction in these errors necessitates an increase in computational resolution, to resolve coastal processes while still covering large domains [13]. An important step towards this direction has been the use of unstructured meshes for the spatial discretisation [14–17].

In the ocean circulation modelling community, the wide range of spatial and temporal scales has motivated the development of spatially adaptive schemes, as a means of local and anisotropic dynamical mesh refinement. Various techniques have been developed, with examples including the structured tree-based hierarchical finite volume Gerris [18] model and the unstructured finite element Fluidity [19] model. The first effort to apply these techniques to the energy balance equation was made by Popinet et al. [20] who combined the adaptive solver of Gerris with WAVEWATCHIII to develop a spatially adaptive spectral wave model. In their work they showed a decrease of one to two orders of magnitude in run-times for practical spatial resolutions. More recently Meixner [21] was the first to apply adaptivity in phase space. By developing a discontinuous finite element spectral wave model, p-adaptivity was applied both in geographic and spectral space. Adjusting the order of the finite element expansions gave significant speed-ups compared to using uniform higher order expansions, in a deep water propagation test case.

This work focuses on applying adaptivity for the refinement of the angular resolution. It is not easy to quantify the directional distribution of ocean waves in a general framework. Observations, however, show that the directional distribution tends to be sharp around the peak frequency [22,23]. As waves propagate outside of their generation area, direction-dispersion further enhances this. Thus, in many cases the energy spectrum only contains energy in a narrow band of directions. (This is even more obvious in coastal areas where waves appear to come from a single direction.) Viewed in this perspective, uniform angular resolutions in spectral wave models are inefficient since for a specific point not all angles have non-zero energy. The adaptive approach proposed here attempts to deal with this problem though the use of compactly supported wavelet basis functions. These can locally resolve details in the angular dimension resulting in a different angular mesh for each computational point.

Wavelets, became an active field of research in the 1980s, with the works of researchers such as Morlet, Grossman and Daubechies [24] on signal processing. Starting as an alternative to Fourier analysis, their popularity soon expanded, owing mainly to the localised nature of wavelet basis in frequency and time, as well as their hierarchical structure. This meant that a localised wavelet transform could be performed with a variable-resolution reconstruction of a signal, which is ideal for applications such as data and image compression [25–27]. These advantages soon drew the attention of the numerical modelling community, as the aforementioned properties provided an efficient framework for adaptive algorithms. Since then, wavelets have been applied to various fields of numerical analysis, including turbulence modelling [28] and partial differential equations such as the Navier–Stokes [29,30], hyperbolic [31,32] and parabolic systems [33,34]. A more comprehensive list of wavelets used in PDE's can be found in [35].

Of more relevance to this work, is the use of wavelets for the discretisation of the Boltzmann transport equation, which provides a natural framework for spectral wave modelling. Both the Boltzmann transport (in non-scattering media) and the energy balance equations are multi-dimensional hyperbolic systems, dealing with the propagation of an energy flux in geographic and phase space [36]. It is worth noting that the energy balance equation is also known as the radiative transfer equation. In the case where only four-wave interactions are considered for the source terms it is also known as the Boltzmann equation [37, p. 30]. Buchan et al. [38] first applied linear and quadratic wavelets for resolving the angular dependence of the Boltzmann transport equation, and then went on to show how they can be used for the application of angular adaptivity [39]. Goffin et al. [40] then extended this to apply goal-based measures to the error metrics driving adaptivity.

In this paper Haar wavelets (named after Alfred Haar) – piecewise constant, hierarchical, compactly supported basis functions – are used for the angular discretisation of the action balance equation and the application of anisotropic angular adaptivity. Haar wavelets are chosen for their simplicity, as well as the fact that they produce sparse system matrices compared to higher order wavelet expansions. For a historical background and an overview of wavelets in general and Haar wavelets in particular the reader can refer to [41], while a more rigorous mathematical background and review of their numerical applications can be found in [42].