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A partially reflecting random walk on spheres algorithm for electrical impedance tomography

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ABSTRACT

In this work, we develop a probabilistic estimator for the voltage-to-current map arising in electrical impedance tomography. This novel so-called *partially reflecting random walk on spheres estimator* enables Monte Carlo methods to compute the voltage-to-current map in an embarrassingly parallel manner, which is an important issue with regard to the corresponding inverse problem. Our method uses the well-known random walk on spheres algorithm inside subdomains where the diffusion coefficient is constant and employs replacement techniques motivated by finite difference discretization to deal with both mixed boundary conditions and interface transmission conditions. We analyze the global bias and the variance of the new estimator both theoretically and experimentally. Subsequently, the variance of the new estimator is considerably reduced via a novel control variate conditional sampling technique which yields a highly efficient hybrid forward solver coupling probabilistic and deterministic algorithms.

1. Introduction

The mathematical formulation of static electrical impedance tomography (EIT) leads to a nonlinear and ill-posed inverse problem, which is unstable with respect to measurement and modeling errors. Namely, the reconstruction of the real-valued conductivity κ in the elliptic *conductivity equation*

$$\nabla \cdot (\kappa \nabla u) = 0 \quad \text{in } D \tag{1}$$

from boundary measurements of the electric potential u and the corresponding current on the boundary of a bounded, convex domain $D \subset \mathbb{R}^d$, $d = 2, 3$, with piecewise smooth boundary ∂D and connected complement. Due to the limited capabilities of static EIT, many practical applications focus on the detection of conductivity anomalies in a known background conductivity rather than conductivity imaging, cf., e.g., Pursiainen [34] and the recent works [39,38] by the second author. In this work, we consider such an *anomaly detection problem*, where a perfectly conducting inclusion occupies a region T inside the domain D. A possible practical application modeled by this setting is breast cancer detection, where the electric conductivity of high-water-content tissue, such as malignant tumors, is approximately one order of magnitude higher than the conductivity of low-water-content tissue, such as fat, which is the main component of healthy breast tissue, cf. [7].

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The most accurate mathematical forward model for real-life impedance tomography is the *complete electrode model* (CEM), cf. [40], where the electric potential u is assumed to satisfy the Robin boundary condition

$$z\nu \cdot \kappa \nabla u|_{\partial D} + u|_{\partial D} = \phi \quad \text{on } \partial D. \tag{2}$$

Here ν denotes the outer unit normal vector on ∂D and the positive constant z is the so-called *contact impedance* which accounts for electrochemical effects at the electrode–skin interface. Given the full Robin-to-Neumann map

$$R_{z,\kappa} : \phi \mapsto \nu \cdot \kappa \nabla u|_{\partial D},$$

that maps the potential on the boundary to the corresponding current across the boundary, this knowledge uniquely determines z and is hence equivalent to the knowledge of the Dirichlet-to-Neumann map. In this case, uniqueness of solutions to the inverse conductivity problem for isotropic conductivities has been proved under various assumptions on both, spatial dimension and regularity of the conductivity, cf., e.g., the works by Astala and Päivärinta [2] for $d = 2$ and Haberman and Tataru [16] for $d = 3$.

Notice that the operator $R_{z,\kappa}$ corresponds to idealized measurements on the whole boundary ∂D. In practice, however, only a finite number of finite-sized electrodes is available and thus only incomplete and noisy measurements of the Robin-to-Neumann map can be obtained. Given such discrete *voltage-to-current maps*, the use of a regularization strategy is mandatory because of the severe ill-posedness of the inverse problem, cf. Alessandrini [1]. In statistical inversion theory, the inverse problem is therefore formulated in the framework of Bayesian statistics, that is, all the variables included in the mathematical model are treated as random variables. The solution to the statistical inverse problem is then given by the posterior probability distribution of the unknown parameters conditioned on the measured data, see, e.g., [17,18]. Computing the *conditional mean* estimate as well as common spread estimates from the posterior density leads to high-dimensional integration problems and Markov chain Monte Carlo (MCMC) techniques are usually employed for this task. However, each sampling step in such an algorithm requires solving the forward problem (1), (2) numerically so that the computation time can easily become excessive. This effect is amplified by the fact that the Robin boundary condition (2) leads to singularities of the solution u at the end points of the electrodes such that numerical approximations, both via finite element and boundary element methods, require very fine discretization.

In this work, we are concerned with the forward problem of EIT. More precisely, we develop a probabilistic estimator for the voltage-to-current map which has potential to overcome the aforementioned drawback if it is used on massively parallel hardware, such as GPUs, within the so-called Bayesian *modeling error approach*, cf. Kaipio and Somersalo [17]. The main advantage of the proposed method, beside its inherent parallel scalability, comes from the fact that the error estimates required for the Bayesian modeling error approach may be computed adaptively and on the fly at almost no additional computational cost. On top of that, our approach is well suited for uncertainty quantification in problems with random parameters.

Due to the advent of multicore computing architectures, probabilistic estimators for the numerical solution of boundary value problems for PDE in three or more dimensions have become a valuable alternative to deterministic methods. This is particularly true, when one needs to compute the solution at only a few points, or when moderate accuracy is sufficient. For instance in biophysical applications, where the linearized Poisson–Boltzmann equation must be solved, efficient probabilistic numerical algorithms have been developed recently, see, e.g., [29,4]. However, in contrast to these works, the derivation of a probabilistic estimator for the voltage-to-current map corresponds to the approximation of paths of the *partially reflecting Brownian motion*, cf. [15], rather than the killed Brownian motion. The partially reflecting Brownian motion behaves like the standard Brownian motion inside the domain and it is prevented from leaving the domain either by absorption or by instantaneous reflection. Under quite general assumptions, a Feynman–Kac type representation formula in terms of the boundary local time process of the partially reflecting Brownian motion for the electric potentials in EIT was recently obtained by Piiroinen and the second author in [33]. It is, however, well-known that direct simulation of the underlying Lebesgue–Stieltjes integrals with respect to the boundary local time process is quite a difficult task, see, e.g., [8,12,13]. To be precise, the first order convergence obtained by Gobet's *half-space approximation* scheme [13] is currently the state of the art in time-discretization methods. See also the recent work [42] by Zhou, Cai and Hsu.

In this work we propose a different approach, namely we discretize with respect to space by expressing the unknown electrical potential as the expectation of some auxiliary random variable obtained via a local finite difference discretization. This yields a novel second order space discretization scheme. A similar technique, using a first order approximation, was first introduced by Mascagni and Simonov in [29] in the context of simulation of diffusion processes in discontinuous media. Also for the simulation of diffusion processes in discontinuous media, second order schemes were proposed and analyzed by Bossy et al. [4] and by Lejay and the first author [22]. The idea to use a local finite difference discretization for the simulation of the boundary behavior of reflecting diffusion processes was introduced recently by the first author and Tanré [26] and further developed by the first author and Nguyen [27]. Other, related schemes were defined by Lejay and Pichot [23] and Lejay and the first author [25]. The main advantage of the method proposed in this work, in comparison to the aforementioned works, lies in the fact that the variance of our estimator is greatly reduced due to a combined control variates conditional sampling technique. Therefore, we expect the resulting hybrid method to become a valuable alternative to established deterministic methods for the problem at hand.

The rest of the paper is structured as follows: We start in Section 2 by describing briefly the modeling of electrode measurements and the anomaly detection problem in EIT. In Section 3 we recall the basic idea of the random walk on spheres