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Efficient model reduction of parametrized systems by matrix discrete empirical interpolation

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ABSTRACT

In this work, we apply a Matrix version of the so-called Discrete Empirical Interpolation (MDEIM) for the efficient reduction of nonaffine parametrized systems arising from the discretization of linear partial differential equations. Dealing with affinely parametrized operators is crucial in order to enhance the online solution of reduced-order models (ROMs). However, in many cases such an affine decomposition is not readily available, and must be recovered through (often) intrusive procedures, such as the empirical interpolation method (EIM) and its discrete variant DEIM. In this paper we show that MDEIM represents a very efficient approach to deal with complex physical and geometrical parametrizations in a non-intrusive, efficient and purely algebraic way. We propose different strategies to combine MDEIM with a state approximation resulting either from a reduced basis greedy approach or Proper Orthogonal Decomposition. A posteriori error estimates accounting for the MDEIM error are also developed in the case of parametrized elliptic and parabolic equations. Finally, the capability of MDEIM to generate accurate and efficient ROMs is demonstrated on the solution of two computationally-intensive classes of problems occurring in engineering contexts, namely PDE-constrained shape optimization and parametrized coupled problems.

1. Introduction

Projection-based model reduction techniques reduce the computational cost associated with the solution of parameter-dependent high-fidelity models by restricting the solution space to a subspace of much smaller dimension. Such techniques have been successfully applied in domains ranging from computational fluid mechanics [1–3], aeroelasticity [4,5] and computational haemodynamics [6] to circuit simulation [7], finance [8] and acoustics [9]. The low cost associated with the solution of reduced order models (ROMs) has in turn allowed their use to accelerate real-time analysis [5,10], PDE-constrained optimization [6,11–15] and uncertainty quantification [16,17] problems. In all these cases – and more generally whenever interested to solve parametrized PDEs many times, for several parametric instances – a suitable offline/online stratagem becomes mandatory to gain a strong computational speedup. Indeed, expensive computations should be carried out in the offline phase, thus leading to a much cheaper online phase. In this context however, the (possibly) complex parametric dependence of the discretized PDE operators have a major impact on the computational efficiency. In the present work, we develop an offline/online procedure that alleviates the computational burden associated with such complex (and

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in particular *nonaffine*) parametric dependencies in the case of linear PDEs. To this end, an affine approximation of the linear operators is developed in the offline phase, leading to inexpensive evaluation of the approximated operators online. These latter are then computed in a general, black-box, purely algebraic way.

1.1. A key problem: dealing with nonaffinities in parametrized ROMs

For the sake of illustration, we consider a linear parametrized dynamical system arising from the spatial discretization of a linear parabolic PDE such as the unsteady advection–diffusion equation

$$\mathbf{M}(t;\boldsymbol{\mu})\frac{d\mathbf{u}_h}{dt}+\mathbf{A}(t;\boldsymbol{\mu})\mathbf{u}_h=\mathbf{g}(t;\boldsymbol{\mu}), \tag{1}$$

where $\mathbf{u}_h=\mathbf{u}_h(t;\boldsymbol{\mu})$, $\mathbf{g}\in\mathbb{R}^n$, $\mathbf{A}\in\mathbb{R}^{n\times n}$ is the matrix encoding the differential operator, $\mathbf{M}\in\mathbb{R}^{n\times n}$ is the mass matrix, while $\boldsymbol{\mu}\in\mathcal{D}\subset\mathbb{R}^P$ denotes the parameter vector. For the sake of completeness, we also consider in this paper the case of a linear parametrized system arising from the spatial discretization of a linear elliptic PDE such as the Helmholtz equation

$$\mathbf{A}(\boldsymbol{\mu})\mathbf{u}_h=\mathbf{g}(\boldsymbol{\mu}). \tag{2}$$

Throughout the paper, we will refer to problems (1) and (2) as high-fidelity or full-order models (FOMs). The idea of a projection-based ROM is to approximate the unknown $\mathbf{u}_h$ in a basis $\mathbf{V}\in\mathbb{R}^{n\times N}$ of reduced dimension $N\ll n$, i.e. $\mathbf{u}_h(t;\boldsymbol{\mu})\approx\mathbf{V}\mathbf{u}_N(t;\boldsymbol{\mu})$, and to enforce the orthogonality of the residual to a test subspace spanned by a suitable basis $\mathbf{W}\in\mathbb{R}^{n\times N}$

$$\mathbf{W}^T\left(\mathbf{M}(t;\boldsymbol{\mu})\mathbf{V}\frac{d\mathbf{u}_N}{dt}+\mathbf{A}(t;\boldsymbol{\mu})\mathbf{V}\mathbf{u}_N-\mathbf{g}(t;\boldsymbol{\mu})\right)=\mathbf{0}.$$

This results in a Petrov–Galerkin projection. Thus, referring for the sake of generality to problem (1), by means of a ROM we seek $\mathbf{u}_N(t;\boldsymbol{\mu})\in\mathbb{R}^N$ such that

$$\mathbf{M}_N(t;\boldsymbol{\mu})\frac{d\mathbf{u}_N}{dt}+\mathbf{A}_N(t;\boldsymbol{\mu})\mathbf{u}_N=\mathbf{g}_N(t;\boldsymbol{\mu}), \tag{3}$$

where the reduced matrices and vectors are given by

$$\mathbf{M}_N(t;\boldsymbol{\mu})=\mathbf{W}^T\mathbf{M}(t;\boldsymbol{\mu})\mathbf{V},\qquad \mathbf{A}_N(t;\boldsymbol{\mu})=\mathbf{W}^T\mathbf{A}(t;\boldsymbol{\mu})\mathbf{V},\qquad \mathbf{g}_N(t;\boldsymbol{\mu})=\mathbf{W}^T\mathbf{g}(t;\boldsymbol{\mu}).$$

The efficient evaluation of the reduced matrices and vectors as time and parameters vary is still one of the main challenges in order to achieve efficient offline construction and online resolution of the ROM.

In the particular case that the system matrices (resp. vectors) can be expressed as an affine combination of constant matrices (resp. vectors) weighted by suitable parameter dependent coefficients, each term of the weighted sum can be projected offline onto the reduced basis (RB) space spanned by $\mathbf{V}$. For instance, let us assume that the matrix $\mathbf{A}(t;\boldsymbol{\mu})$ admits an affine decomposition

$$\mathbf{A}(t;\boldsymbol{\mu})=\sum_{q=1}^{M}\theta_q(t;\boldsymbol{\mu})\mathbf{A}_q. \tag{4}$$

Then

$$\mathbf{A}_N(t;\boldsymbol{\mu})=\mathbf{W}^T\mathbf{A}(t;\boldsymbol{\mu})\mathbf{V}=\sum_{q=1}^{M}\theta_q(t;\boldsymbol{\mu})\,\mathbf{W}^T\mathbf{A}_q\mathbf{V},$$

where $\theta_q\colon[0,T]\times\mathcal{D}\mapsto\mathbb{R}$ and $\mathbf{A}_q\in\mathbb{R}^{n\times n}$ are given functions and matrices, respectively, for $q=1,\dots,M$. Since the reduced matrices $\mathbf{W}^T\mathbf{A}_q\mathbf{V}\in\mathbb{R}^{N\times N}$ can be precomputed and stored offline, the online construction of the ROM for a given $(t;\boldsymbol{\mu})$ is fast and efficient as long as $M\ll n$.

On the other hand, if $\mathbf{A}(t;\boldsymbol{\mu})$ is not affine – that is, if it does not feature an affine decomposition (4) – this computational strategy breaks down. In this case, the construction of the ROM for a given $(t,\boldsymbol{\mu})$ requires first to assemble the full-order matrices and vectors and then to project them onto the reduced space, thus entailing a computational complexity which scales with the dimension of the large scale system. Unfortunately, many applications of interest feature a nonaffine dependency with respect to time and/or parameters. This is the case, for instance, when dealing with parametrized shape deformations, multi-physics problems and nonlinear PDEs linearized around a steady-state. Furthermore, even if the problem admits an affine decomposition, exploiting this decomposition might require intrusive changes to the high-fidelity model implementation (see, e.g., [18]), or even be impossible when using black-box high-fidelity solvers. As a matter of fact, we would like to reuse during the offline stage a computational code already designed for the solution of the high-fidelity problem at hand, and then to build a ROM by relying on the minimum amount of algebraic structures of the full-order model. In order to recover the affine structure (4) in those cases where the operator $\mathbf{A}(t;\boldsymbol{\mu})$ is nonaffine (or (4) is not readily available), we must introduce a further level of reduction, called hyper-reduction or system approximation [19,2], employing suitable techniques which are briefly reviewed in the following section.