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Spectral direction splitting methods for two-dimensional space fractional diffusion equations ☆

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A numerical method for a kind of time-dependent two-dimensional two-sided space fractional diffusion equations is developed in this paper. The proposed method combines a time scheme based on direction splitting approaches and a spectral method for the spatial discretization. The direction splitting approach renders the underlying two-dimensional equation into a set of one-dimensional space fractional diffusion equations at each time step. Then these one-dimensional equations are solved by using the spectral method based on weak formulations. A time error estimate is derived for the semi-discrete solution, and the unconditional stability of the fully discretized scheme is proved. Some numerical examples are presented to validate the proposed method.

1. Introduction

The fractional partial differential equations are now winning more and more scientific applications across a variety of fields including control theory, biology, electrochemical processes, porous media, viscoelastic materials, polymer, finance, etc. The universality of anomalous diffusion phenomenon in various experiments has led to an intensive investigation of these equations in recent years. The fractional diffusion equation considered in this paper is of interest not only in its own right, but also in that it constitutes the principal part in solving many other more general fractional differential equations. We refer, e.g., to [19,20] for modeling chaotic dynamics charge transport in amorphous semiconductors, [18] for nuclear magnetic resonance diffusometry in disordered materials, and [15] for modeling the propagation of mechanical diffusive wave in viscoelastic media.

There have been a number of numerical methods constructed for the time-fractional diffusion equations; see, e.g., [13] for a finite difference scheme in time and spectral method in space, [34] for a particle tracking approach, [10] for a time–space spectral method, [35] for an alternating direction implicit scheme, [22] for finite difference schemes for a variable-order equation, [9] for a finite element method, and [32] for a spectral method using Jacobi polyfractonomials for fractional ODEs.

On the other hand, the space-fractional diffusion equations have also been a subject of many investigations. Among the existing numerical methods for this kind of fractional diffusion equations, we mention the finite difference methods based on the shifted Grüwald formulae in [16,17,23], spline approximations [21], the finite difference methods for Riesz fractional derivatives [14,31], the spectral method based on weak formulation [11], a finite element method for the space and

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time fractional Fokker–Planck equation [4], a Runge–Kutta discontinuous Galerkin methods for one- and two-dimensional fractional diffusion equations [8], a finite difference/element method for a two-dimensional modified fractional diffusion equation [33], and a method combining the alternating direction implicit method and the Crank–Nicolson scheme [3]. Wang and Wang [26] developed, without giving a stability analysis, an alternating direction implicit finite difference method for space-fractional diffusion equations.

In this paper we aim at designing an efficient method for solving the space fractional diffusion equation. The proposed method combines a stable direction splitting scheme with a spectral discretization in space that allows for efficient implementation. This work was motivated by the attempt to take double advantages of the spectral method and the direction splitting approach. Firstly, the fractional diffusion equation is featured by the presence of non-local operators involved in the definition of fractional derivatives. These non-local operators make any approximation, either low or high order methods, into non-sparse linear system. This nature obviously reduces the advantage of low order methods in term of computational complexity, and favours the use of high order methods if the solution to be approximated is smooth enough. It is well known that, as compared to low order methods, higher order methods like spectral methods require less degrees of freedom to achieve the same accuracy. This consideration has inspired a recent series of papers [10–12,32], which focused on developing spectral methods for some time/space fractional differential equations. It is worthy to mention that Wang et al. [28,29] showed that a fractional equation with smooth data can have non-smooth solutions. Hence, how to guarantee the smoothness of the solution is a difficult issue. Secondly, despite of the efficiency of the spectral method, the numerical solution of the fractional diffusion equation in high dimension requires more numerical techniques. Direction splitting methods are considered as powerful techniques which allow to split the underlying high dimensional problem into a set of one-dimensional sub-problems, thus can considerably reduce the computational complexity for some traditional equations; see, e.g., [2,7]. Note that Wang et al. [27,25] constructed and analyzed finite difference/ADI methods for fractional diffusion equations with variable coefficients. Their methods have also been shown to be fast with efficient storage.

The main purpose of this paper is to develop a stable direction splitting scheme in time with a spectral discretization in space for the space fractional diffusion equation. The stability of the overall scheme is rigorously established. Although such a combination has been constructed and analysed for a number of traditional equations, it's extension to problems with fractional operators does not seem to be trivial.

The outline of this paper is as follows. In the next section we describe the underlying problem, and construct the direction splitting scheme. A splitting error estimate is derived. In Section 3, we propose the full discrete scheme by using a spectral method for the spatial discretization of the fractional differential operators, and carry out a detailed analysis for the stability of the proposed scheme. The unconditional stability is proved under an assumption on the diffusion coefficients. We give in Section 4 some implementation details and present the numerical results to verify the stability and accuracy of the method. Finally, we give some concluding remarks in Section 5.

2. Direction splitting scheme

We consider the following two-dimensional space fractional diffusion equation:

$$\frac{\partial u(x,y,t)}{\partial t} = Lu(x,y,t) + f(x,y,t), \tag{2.1}$$

where $t \in (0,T]$, $(x,y) \in \Omega = \Lambda^2$, $\Lambda = (-1,1)$, $f(x,y,t)$ is a source function. L is the fractional operator defined by

$$Lu(x,y,t) = p\big(D_x^\alpha u(x,y,t) + {}_xD^\alpha u(x,y,t)\big) + q\big(D_y^\beta u(x,y,t) + {}_yD^\beta u(x,y,t)\big), \tag{2.2}$$

with p and q being positive diffusion coefficients, and the fractional derivatives of order α or β with $1 < \alpha, \beta < 2$ being defined in the Riemann–Liouville sense as follows:

$$D_x^\gamma \varphi(x) = \frac{1}{\Gamma(2-\gamma)} \frac{d^2}{dx^2} \int_{-1}^{x} \frac{\varphi(\xi)d\xi}{(x-\xi)^{\gamma-1}}, \ \forall x \in \Lambda, \gamma = \alpha, \beta, \tag{2.3}$$

$${}_xD^\gamma \varphi(x) = \frac{1}{\Gamma(2-\gamma)} \frac{d^2}{dx^2} \int_{x}^{1} \frac{\varphi(\xi)d\xi}{(\xi-x)^{\gamma-1}}, \ \forall x \in \Lambda, \gamma = \alpha, \beta. \tag{2.4}$$

Usually D_x^γ is called the left-sided fractional derivative, and ${}_xD^\gamma$ the right-sided fractional derivative of order γ.

The equation (2.1) is subject to the following initial and boundary conditions:

$$u(x,y,0) = u_0(x,y), \ \forall (x,y) \in \Omega, \tag{2.5}$$

$$u(x,y,t)|_{\partial\Omega} = 0, \ \forall t \in (0,T]. \tag{2.6}$$

The sum of terms $D_x^\alpha u(x,y,t) + {}_xD^\alpha u(x,y,t)$ in (2.2) is sometimes denoted by $D_{|x|}^\alpha u(x,y,t)$, called symmetrized fractional derivative. It has been shown in [11] that the existence and uniqueness of a solution to (2.1)–(2.5)–(2.6) can be guaranteed by keeping only one derivative term in x-direction and one derivative term in y-direction in the right-hand side