Journal of Computational Physics 298 (2015) 1–28

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

A boundary integral algorithm for the Laplace Dirichlet–Neumann mixed eigenvalue problem

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ARTICLE INFO

Article history:
Received 1 November 2014
Accepted 13 May 2015
Available online 3 June 2015

Keywords:
Fourier continuation
Boundary integral operators
Mixed boundary conditions
Zaremba eigenvalue problem
Laplace eigenvalue problem

ABSTRACT

We present a novel integral-equation algorithm for evaluation of *Zaremba eigenvalues and eigenfunctions*, that is, eigenvalues and eigenfunctions of the Laplace operator with mixed Dirichlet–Neumann boundary conditions; of course, (slight modifications of) our algorithms are also applicable to the pure Dirichlet and Neumann eigenproblems. Expressing the eigenfunctions by means of an ansatz based on the single layer boundary operator, the Zaremba eigenproblem is transformed into a nonlinear equation for the eigenvalue μ. For smooth domains the singular structure at Dirichlet–Neumann junctions is incorporated as part of our corresponding numerical algorithm—which otherwise relies on use of the cosine change of variables, trigonometric polynomials and, to avoid the Gibbs phenomenon that would arise from the solution singularities, the Fourier Continuation method (FC). The resulting numerical algorithm converges with high order accuracy without recourse to use of meshes finer than those resulting from the cosine transformation. For non-smooth (Lipschitz) domains, in turn, an alternative algorithm is presented which achieves high-order accuracy on the basis of graded meshes. In either case, smooth or Lipschitz boundary, eigenvalues are evaluated by searching for zero minimal singular values of a suitably stabilized discrete version of the single layer operator mentioned above. (The stabilization technique is used to enable robust non-local zero searches.) The resulting methods, which are fast and highly accurate for high- and low-frequencies alike, can solve extremely challenging two-dimensional Dirichlet, Neumann and Zaremba eigenproblems with high accuracies in short computing times—enabling, in particular, evaluation of thousands of eigenvalues and corresponding eigenfunctions for a given smooth or non-smooth geometry with nearly full double-precision accuracy.

1. Introduction

This paper presents a novel boundary integral strategy for the *numerical solution of the Zaremba eigenproblem*, that is, numerical approximation of eigenvalues λ_j, $j = 1, 2, \ldots$ and associated eigenfunctions $u_j \in H^1(\Omega)$ of the Laplace operator under mixed Dirichlet–Neumann boundary conditions (cf. equation (1) below); naturally, the main elements of our algorithms are also applicable to the pure Dirichlet and Neumann eigenproblems.

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http://dx.doi.org/10.1016/j.jcp.2015.05.016
0021-9991/

The use of boundary integral equations for the solution of Laplace eigenproblems has been explored in a number of contributions, including methods based on collocation [14,24] and Galerkin [39,40] boundary element approaches for the Dirichlet and Neumann problems. The boundary element strategy for three-dimensional Dirichlet eigenproblems presented in [39,40], for example, yields errors that decrease cubically with the spatial mesh-sizes. However, as mentioned in [39], "the convergence regions for the eigenvalues are still local" and "other techniques have to be considered and analyzed in order to increase the robustness". Focusing on two-dimensional Laplace eigenvalue problems, in this paper we present a Nyström algorithm that can achieve any user-prescribed order of convergence for smooth and non-smooth domains alike, as well as a novel, robust, search algorithm that yields fast eigenvalue convergence from nonlocal initial guesses—see Section 6 for details. To the best of our knowledge, further, the present algorithm is the first boundary-integral method for eigenvalue problems of Zaremba type.

Integral equation formulations for eigenvalue problems are advantageous as they 1) Result in a reduction in the problem dimensionality; and, as described in Section 4, they 2) Greatly facilitate efficient treatment of the eigenfunction singularities that occur around corners and Dirichlet–Neumann transition points. As a counterpart, however, the integral form of the eigenvalue problem (cf. equation (4) below) is nonlinear (since the eigenvalue appears as part of the integral kernel), and eigenvalues and eigenfunctions must therefore be found by means of an appropriate nonlinear equation solver.

It is important to note that the eigenfunctions in equation (1) as well as the corresponding densities ψ in (6) exhibit singularities at corners and Dirichlet–Neumann junctions. In particular, in contrast to the situation for the pure Dirichlet or Neumann eigenfunctions, even for a smooth boundary Γ the eigenfunctions of (1) are singular: they are elements of $H^1(\Omega)$ but not of $H^2(\Omega)$. The specific asymptotic forms of these singularities for both smooth and Lipschitz domains are described in Section 4.

In Section 5 we discuss novel discretization strategies for our integral formulation of the Zaremba eigenvalue problem which yield high order accuracy in spite of the poor regularity of eigenfunctions and densities near Dirichlet–Neumann junctions. In the smooth domain case our Zaremba eigensolver includes an adaptation of the novel Fourier Continuation (FC) method [3,8,30] (which accurately expresses non-periodic functions in terms of Fourier series; see Section 5.2) and it explicitly incorporates the asymptotic behavior of solutions near the Dirichlet–Neumann junction. For possibly non-smooth curves Γ, on the other hand, an approach is introduced in Section 5.3 which, on the basis of a graded-mesh discretizations [19,25,27,31,38], yields once again high-order accuracy—albeit not as efficiently, for smooth domains, as that resulting from the FC-based algorithm (cf. Remark 5.7 and Section 8.5).

A method for solution of Dirichlet eigenproblems for the Laplace operator that, like ours, is based on detection of parameter values for which a certain matrix is not invertible, was introduced in [22]. In that early contribution this *Method of Particular Solutions* (MPS) (which approximates eigenfunctions as linear combinations of Fourier–Bessel functions) performs the singularity search via a corresponding search for zeroes of the matrix determinant. Subsequently, [35] substituted this strategy by a search for zeroes of minimum singular values—an idea which, with some variations, is incorporated as part of the algorithm proposed presently as well. A modified version of the MPS, which was introduced in Ref. [4], alleviates some difficulties associated with the conditioning of the method.

As it happens, however, a direct evaluation of the zeroes of the smallest singular value $\eta_n(\mu)$ of our $n \times n$ discretized boundary integral operator is highly challenging. Indeed, as shown in Section 6, the function $\eta_n(\mu)$ is essentially constant away from its roots, and therefore descent-based approaches such as the Newton method fail to converge to the roots of η_n unless an extremely fine mesh of initial guesses is used. A modified integral equation formulation (with associated smallest singular values $\widetilde{\eta}_n(\mu)$) is introduced in Section 6 that, on the basis of ideas introduced in [4], successfully tackles this difficulty (cf. Remark 6.3). As demonstrated in Section 8, the resulting eigensolvers, which are fast and highly accurate for high- and low-frequencies alike, can solve extremely challenging two-dimensional Dirichlet, Neumann and Zaremba eigenproblems with high accuracies in short computing times. In particular, as illustrated in Section 8.6, the proposed algorithms can evaluate thousands of Zaremba, Dirichlet or Neumann eigenvalues and eigenfunctions with nearly full double-precision accuracy for both smooth and non-smooth domains. The algorithms presented in this paper can further be generalized to enable evaluation of eigenvalues of multiply-connected domains—for which integral eigensolvers can give rise to spurious resonances [15,16]; a description of the method for multiply connected domains together with a number of illustrative numerical examples are provided in Section 8.7.

The recent contribution [50] relies on determination of zeroes of matrix determinants to address, in the context of the pure Dirichlet eigenvalue problem, certain challenges posed by search methods based on use of smallest singular values—which are generally non-smooth function of μ. As indicated in Remark 6.4, however, a relatively straightforward sign-changing procedure we use yields singular values that vary smoothly (indeed, analytically!) with μ, and thus eliminates difficulties arising from non-smoothness. Note that generalizations of the present methods to algorithms that rely on iterative singular-value computations and fast evaluations of the relevant integral operators [6,10,37] (which should enable solution of higher frequency/three-dimensional problems) can be envisioned.

This paper is organized as follows: Section 2 describes the Laplace–Zaremba eigenvalue problem for a class of domains in $\mathbb{R}^2$ and Section 3 puts forth an equivalent boundary integral formulation based on representation of eigenfunctions via single layer potentials. Section 4 then discusses the singular structure of eigenfunctions and associated integral densities at both smooth and non-smooth Dirichlet–Neumann junctions; these results are incorporated in the high-order numerical quadratures described in Section 5. Section 6 introduces a certain normalization procedure which leads to an efficient eigenvalue-search algorithm. Once eigenvalues and corresponding integral densities have been obtained, the eigenfunc-